YOUNG CHILDREN’S SPATIAL CONCEPTIONS OF TWO-DIMENSIONAL GRID STRUCTURES

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Abstract

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Spatial reasoning and spatial structures are relatively new areas of research in mathematics education. In this study of children exploring spatial conceptions of grid structures, twenty-one children (ages 4-9) were given a series of tasks involving square grids during virtual interviews. As a result of an ideal-type analysis of the qualitative data, a typology of conceptions of grids emerged showing five distinct categories sequenced from very early conceptions of square grids (as a series of isolated cells) to more coordinated structuring (as related and intersecting rows and columns). The five categories - Single Cell Structuring, Partial Unit Building, Whole Figure and Parts-of-Figure Noticing, Composite Unit Structuring and Coordinated Structuring - are described through illustrative examples. Students' gestures, language and diagrams were considered together when constructing the types. Interestingly, the spatial structure of grids was not readily apparent to many students and in fact was found to be complex for students to conceptualize. With minimal research on grids as a spatial structure in the mathematics education research field, there is strong potential for further investigation in this area.

Key Words: Mathematics; Young children; Spatial reasoning; Grids

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The vast majority of this thesis took place during the COVID-19 pandemic. While there are many areas of this work that will continue on, completing this thesis feels very cathartic in terms of closing a chapter on a very difficult period. Despite the many challenges and the sense of isolation during the pandemic, there was also an overwhelming feeling of support and community that came through this work. I am forever grateful to *all* of my friends and neighbours (I want to list all of you here but there are just too many of you!) who were open to making time and space in their lives to think about grids when it seemed like the least important thing to think about.

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Introduction

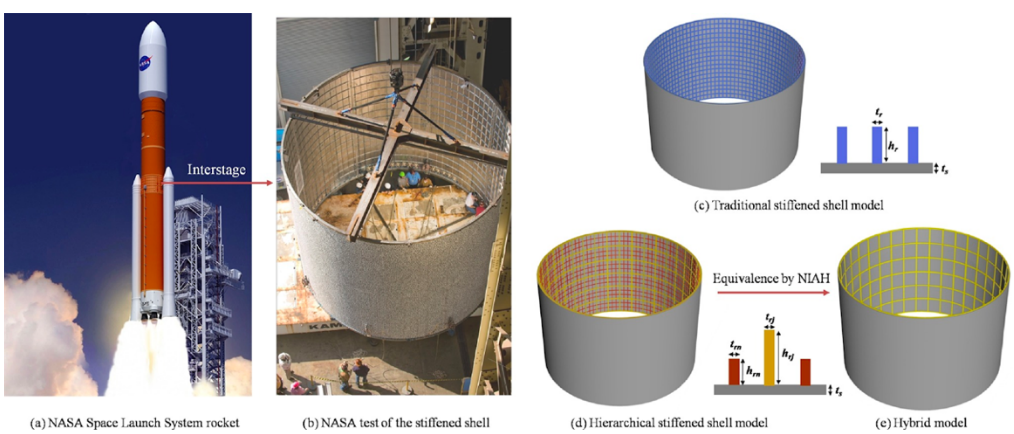
1.1 Introduction and research context

A rectangular array of squares is one type of grid that is a widely used tool in the mathematics classrooms. Grids often underlie mathematics tasks because they help students measure, divide and organize two-dimensional space (e.g., area measurements, multiplication, fractions, mapping, coding etc.). However, grids are also taken-for-granted by educators, and adults in general, in terms of their structure. How young children “see” grids is more complicated than we might realize; there may be a wide range of ways students interpret the fundamental properties of grids. Studying how young children perceive grids links directly to spatial reasoning which is a rapidly growing area of interest in mathematics education research (Bruce et al., 2017).

Whether or not we consciously recognize it, spatial reasoning is threaded throughout our lives, from how we load the dishwasher to how we imagine ourselves in relation to the world around us, to how we mentally manipulate or move objects. All of us engage in spatial reasoning and all of us can improve our spatial skills with training (Newcombe, 2010). Spatial reasoning is foundational to many STEM related fields (Newcombe, 2010). And of course, grids are widely used in many STEM related fields to support the visual modeling of problems. For example, in the field of aerospace technology, researchers have been designing and testing cylindrical shells used in building rockets so that the shells can withstand tremendous pressures and temperatures (see Figure 1) (Wang et al., 2017). In order to improve the structure of the shells, different grids (orthogrid, rotated triangle grid, triangle grid, mixed triangle grid patterns) were imagined, constructed and tested (see Figure 2) (Wang et al., 2017). Grids are not just useful tools for spatial reasoning in the STEM fields. Website developers, interior designers, computer animators and geologists are just a few examples of professions that make use of grids. Grids help us to think spatially as they support us in “seeing” and describing spatial relationships and information as well as transforming and moving objects. Researchers have concluded that “spatial imagery, that is, the ability to represent the spatial relations between objects and to imagine spatial transformations, may be foundational for mathematical thinking” (Frick, 2019, p. 1466). In addition, some researchers posit that grids are foundational to reasoning spatially in mathematics (Battista 1999, Clements et al., 2017, Mulligan et al., 2020, Outhred & Mitchelmore 2000, Sarama et al., 2003).

Figure 1

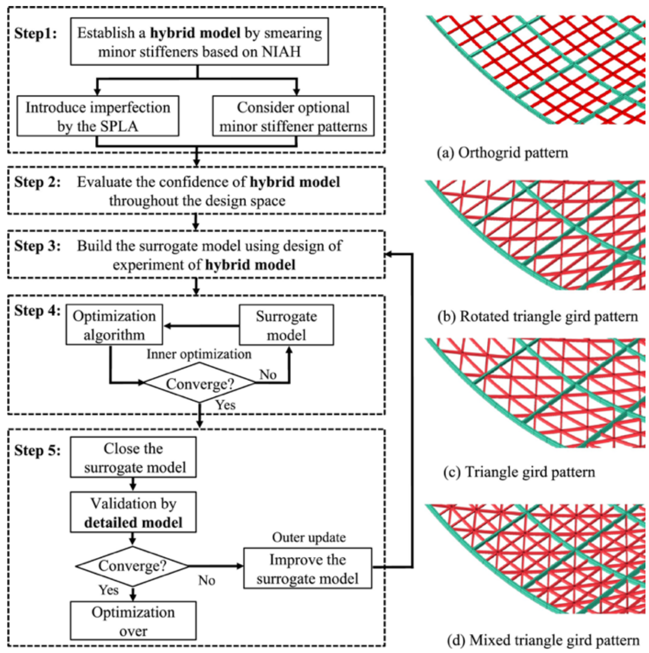
Example of grids as a tool for supporting visual modeling in STEM field



Note. From “Grid-pattern optimization framework of novel hierarchical stiffened shells allowing for imperfection sensitivity,” by Tian K Wang, C. Zhou, P. Hao, Y. Zheng, Y. Ma, and J. Wang, 2017, *Aerospace Science and Technology*, *62*, p. 115 (https://doi.org/10.1016/j.ast.2016.12.002)

Figure 2

Example of the different types of grids tested to optimize the structure of a rocket shell



Note. From “Grid-pattern optimization framework of novel hierarchical stiffened shells allowing for imperfection sensitivity,” by Tian K Wang, C. Zhou, P. Hao, Y. Zheng, Y. Ma, and J. Wang, 2017, *Aerospace Science and Technology*, *62*, p. 117 (https://doi.org/10.1016/j.ast.2016.12.002)

Spatial reasoning has been a minor area of study within mathematics education research since the 1970’s, however it is of renewed interest as a result of research developments, curriculum developments and growing attention to STEM fields (Bruce et al., 2017, Newcombe 2010, Newcombe & Shipley 2015, Sinclair & Bruce 2015). For example, the Spatial Reasoning Study Group (SRSG) is an interdisciplinary and international think tank of researchers who share an interest in the teaching and learning of mathematics (Bruce et al., 2017). This research collaborative has generated an extensive list of examples of spatial reasoning that includes “locating, orienting, decomposing/recomposing, balancing, patterning diagramming, navigating, comparing, scaling, transforming and seeing symmetry” (Bruce et al., 2017, p. 146). Through the generation of this list of actions the group was able to form a preliminary definition of spatial reasoning; “the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects” (Bruce et al., 2017, p. 146). Reflecting on the above list of actions associated with spatial reasoning generated by the SRSG, it is difficult to imagine examples that could not be supported in some way or explored through grids. Broadly speaking, grids embody the intersection of number and space, and as such they are a powerful spatial object for reasoning mathematically (Davis et al., 2015).

We know that children “come to school with a tremendous repertoire of informal spatial understandings that can and should be developed” (Whiteley et al., 2015, p. 8). Fortunately, spatial reasoning has been demonstrated to be a highly malleable and transferable skill (Uttal et al., 2013). The National Research Council explains that “spatial thinking can be learned *and* it can and should be taught at all levels of the education system” (NRC, 2006 p.3). Recently, there has been increased interest and attention towards developing spatial thinking in young children because it supports later mathematics success (Farmer et al., 2013, Hawes et al., 2017, Hawes & Ansari, 2020, Moss et al., 2016, Newcombe 2010, Sinclair & Bruce 2015, Verdine et al., 2014,). Grids, however, remain an underutilized and under examined spatial object within mathematics education and mathematics education research.

What we do know about how students think about grids is largely thanks to the work of researchers who are primarily interested in students' measurement thinking. Battista, Clements, Samara, Mitchelmore and Barrett are educational researchers who have greatly contributed to our understanding of students’ perceptions of grids. As a result of their measurement investigations researchers are uncovering how challenging it can be for children to construct their understanding of a grids spatial structure (Barrett et al., 2017, Battista 1999, Battista et al., 1998, Battista & Clements 1996, Clements et al., 2017, Outhred & Mitchelmore, 1992, Outhred & Mitchelmore 2000, Sarama et al., 2003). This research has been instrumental in mapping out developmental continuums of student perceptions of grids from a measurement perspective (Battista et al., 1998, Clements et al., 2017).

With the above research in mind, the purpose of this thesis was to uncover and describe early conceptions of grids as *spatial objects* in particular. Specifically, this study explored the different ways young children (ages 4-9) made sense of grids with words, drawings and gestures. This required detailed observation of students through task-based interviews.

1.2 Researcher Beliefs

As a classroom teacher for ten years, I had the privilege of observing and supporting a wide range of students (Grades 2-6) in their learning of mathematics. I was often learning alongside my students as I worked to develop the unique content knowledge that educators require to support mathematics learning. As a result of my observations and building my knowledge of current research, I developed a deep interest in understanding the spatial elements that are foundational to mathematics concepts. Watching carefully, I observed students using language, gestures, drawings as well as physical tools as part of how they were thinking about and constructing their understanding. How students learn mathematics has been an endlessly fascinating area of study for me.

I was fortunate to step away from the classroom and support a broader range of students and educators from Kindergarten to Grade 9 in my role as a mathematics consultant. My time in this role gave me the opportunity to test theories I had developed and read about, related to how students learn math across a wide range of grades, classrooms and school communities. Through these experiences I have come to believe that anyone, at any age, given any learning profile, is capable of learning and loving mathematics if given the opportunity to engage playfully with spatially grounded tasks guided by their peers and educators who see them and all of their humanity.

I also recognize that mathematics has had a legacy of exclusionary approaches and a kind of privilege that has separated many children and adults from feeling they can engage in mathematics. I feel passionately that the work we do to uncover the elements of mathematics that are intuitive to young children and that will help to develop their spatial reasoning is in service of making difficult areas of mathematics more accessible to all students. Ultimately, I believe that supporting spatial reasoning in the early years is an underutilized avenue for opening up mathematics for all students.

1.3 Research Questions

The purpose of this study is to explore how young children (ages 4-9) think about grids as spatial objects in mathematics.

The aim of the study is to:

1. How do young children perceive the spatial structure of two-dimensional square grids?
2. What drawings, gestures and language correspond with students' conceptions of square grids?

Literature Review

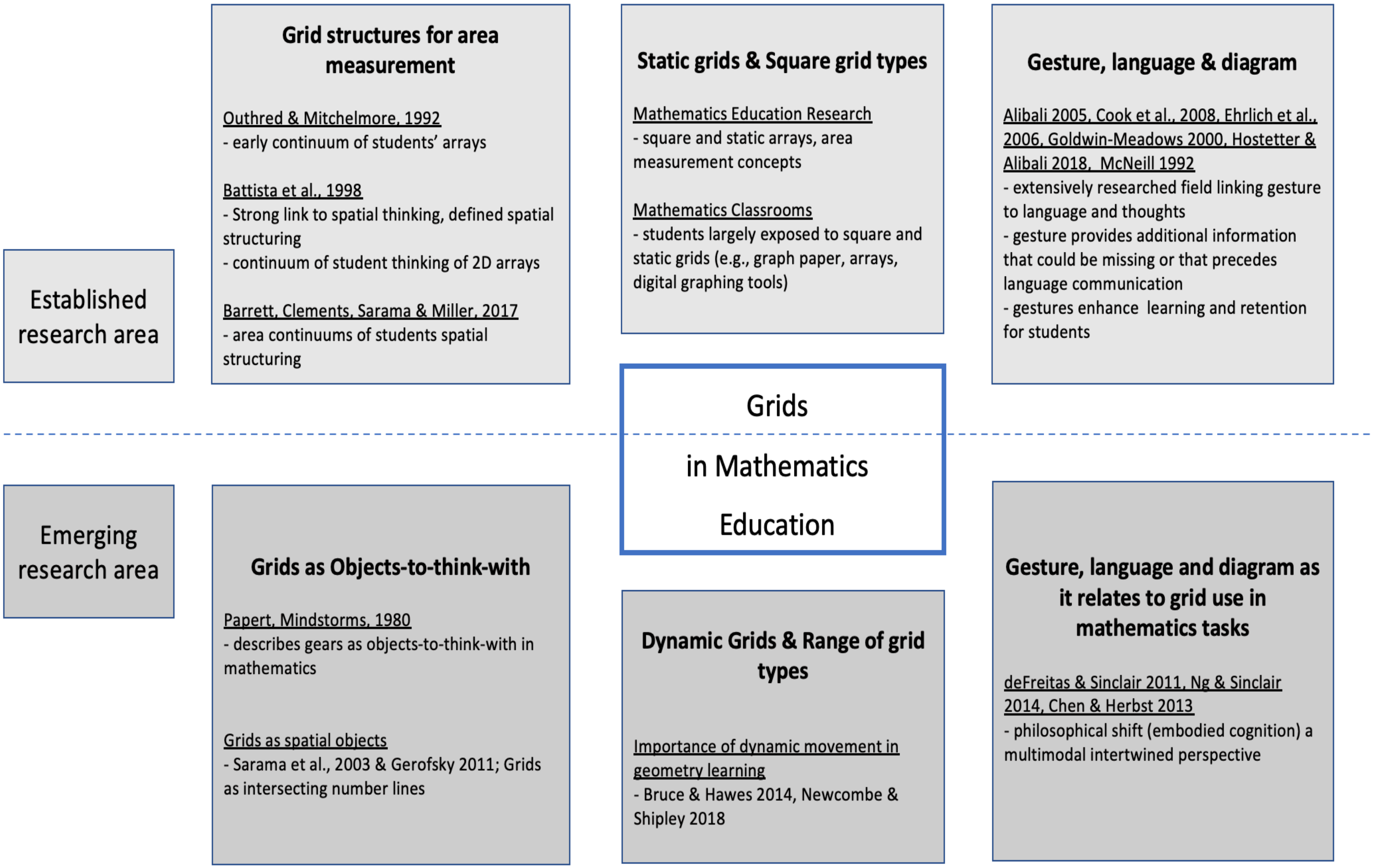
This literature review centres on three related areas that shed light on how young children make sense of the mathematical use of, and their perceptions of grids:

- Grid structures
- Objects-to-think-with
- Gesture, language and diagram as part of spatial reasoning

To begin, I discuss the structure of grids and their use in mathematics education (largely within measurement applications), including their significance as spatial objects. I then discuss how grids as spatial objects have the potential to become objects-to-think-with, a way to see and model the world, construct new learning and even apply them when problem solving. Finally, I highlight some of the research on the role of gestures, language and diagrams as a multimodal expression of how students' are reasoning spatially. Below, Figure 3 provides a framing diagram for the research within this literature review.

Figure 3

Map of the literature



2.1 Grid structures: An overview

In this first section of the literature review, I provide an overview of research related to grids as widely used structures in mathematics. There is limited research on grid structures in education, but a cluster of researchers have indeed studied developmental continua related to grids, mostly within the frame of area measurement conceptions. The main researchers who have contributed to spatial structuring research include Battista et al., 1998, Clements et al., 2017, Mulligan et al., 2020, Outhred & Mitchelmore 2000 and Sarama et al., 2003 and they are largely featured in this literature review.

The term “Spatial Structuring” is central to this thesis and to the existing body of research that can be linked to gridwork. Battista (1999) defines spatial structuring as;

> the mental operation of constructing an organization or form for an object or set of objects. It determines the object’s nature, shape, or composition by identifying its spatial components, relating and combining these components, and establishing interrelationships between components and the object. (p. 171)

Essentially, Battista explains that students are looking at an “unstructured world” and applying an element of structure to it (e.g., organization, movement, defining lines, points, shape etc.) as part of how they are “developing their meaningful construction of geometric and spatial ideas” (Battista 1999, p. 177). We will explore this element of structure within grid structures specifically, the use of grids in mathematics classrooms, and research into developmental continua related to measurement. Then we will turn to considering grids as spatial objects and the related implications.

2.1.1 What are grids?

What are grids? Grids are so ubiquitous across the landscape of mathematics that it may seem redundant to stop and consider such a question. Very often it is assumed that learners (even young learners) understand the anatomy of a grid even though they have rarely been explored explicitly in math class. There are different ways to define “grids” depending on the discipline or context of usage (e.g., graphic design, computing, electronics etc.). Within different fields of mathematics, grids are defined and used differently depending on their application. Let’s consider how the different definitions of grids impact our conceptions of their structure.

Grids as tessellations of squares

Consider a grid on a “usual (x, y) coordinate plane and the square in it with vertices at (0,0), (1,0), (1,1) and (0,1), which consists of the points whose coordinates satisfy $0 \leq x \leq 1$, $0 \leq y \leq 1$. This square can be moved horizontally and vertically” (Gowers 2008, p.208). We could continue to replicate the original square until “copies of the square cover the whole plane, with four squares coming together at each point with integer coordinates. The plane is said to be *tiled* or *tessellated* (from the Latin word for a marble chip in a mosaic)” (Gowers 2008, p.208). This visual description draws a lattice structure of squares in the mind's eye, and focuses our attention on the way in which the squares are arranged and arrayed.

Grids as cartesian coordinates and distance

Depending on our purpose, we might instead choose to conceive of grids by considering Cartesian coordinates. This definition of grids asks us to choose;

> an origin and two directions *X* and *Y,* usually at right angles to each other. Then the pair of numbers (*a,b*) stands for the point you reach in the plane if you go the distance *a* in direction *X* and a distance *b* in direction *Y* (where if *a* is a negative number such as -2, this is interpreted as going a distance +2 in the opposite direction to *X,* and similarly for *b*). Another way of saying this is let *x* and *y* stand for the unit vectors in directions *X* and *Y,* respectively, so their Cartesian coordinates are (1,0) and (0,1). Then every point in the plane is a so-called *linear combination a x* + *b y* of the *basis vectors x* and *y.* (Gowers 2008, p.21)

This definition calls our attention to the points of intersection made through the intersecting parallel lines defining the plane. In particular, this description focuses on unit distances between intervals and points to the importance of 'linear combinations'. \

Mutable grids

It is important to note that in both of the above descriptions we are only describing square grids, when in fact, there are many possible grid configurations across two- and three- dimensional planes. Grids can have dynamic vanishing points. They can bend and wrap around objects, be scaled up and down, stretched or twisted. There are also many grid types (e.g., polar grids, triangular grids etc.). Grids are malleable and can be scaled to fit almost any context. They can be laid behind or over top of other objects, stretched, skewed, rotated or even wrapped around another surface. It is taken for granted that grid lines represent unending divisions in continuous space stretching out infinitely (Lakoff & Núñez, 2000).

For the purposes of this thesis, which examines grids within the elementary mathematics classroom, my working definition describes grids as; spatial objects constructed through the patterned repetition of intersecting parallel lines, generating related angles and side lengths within regions and points of intersection; in its dynamic form grid lines can stretch, compress, bend or angle toward a given vanishing point.

2.1.2 Grids in mathematics classrooms

Despite the vast possibilities grids present, the most widely used and often exclusively used representation in the elementary mathematics classroom is a two-dimensional rectangular array of squares. The traditional square grid structure that students encounter throughout their mathematics learning is presented as static in nature, the perspective and vanishing point are therefore fixed.

As illustrated in the previous section, the way in which we choose to define grids impacts our conceptions of their structure. Depending on the context or the usage, students will need to consider grids both as a series of perpendicular lines with distanced intervals and as a tessellation of squares that perfectly connect along their edges. In the mathematics classroom the purpose or application that the students are engaged in will ultimately determine to which elements of a grid's structure students should attend. The sub-sections that follow present brief thought experiments, imagining the ways in which students may experience grids given these two different contexts (as tessellations and as cartesian coordinates).

Grids as tessellations of squares in the mathematics classrooms

In the classroom, educators might expect students to conceive of a rectangular based grid structure as composed of a tessellation of squares across rows and columns or similarly as space that has been partitioned into rows and columns. In this case, the rows and columns have an orthogonal relationship and as such relate to each other in a unique way. Individual cells in a row overlap or converge at their intersections, rows can be partitioned into columns and vice versa. This conception of a grid structure encourages students to attend to the area of the cells and therefore supports thinking about area measurements, multiplication, fractional and proportional relationships. In this conception, grids help us to easily partition, quantify, measure and proportion space.

Grids as cartesian coordinates in the mathematics classrooms

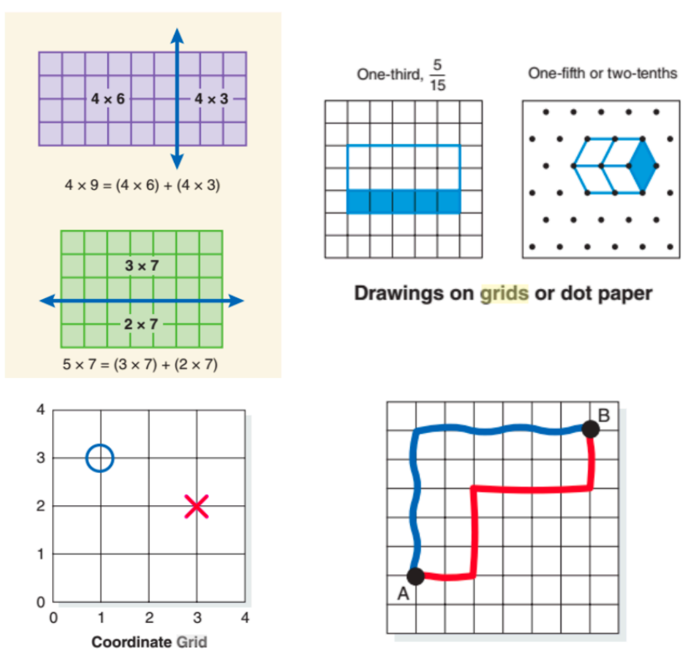
Educators could equally emphasize the intersection of perpendicular lines that compose a rectangular grid structure. Grid lines and points of intersection support us in organizing, locating and mapping space. We can label grid lines, grid cells and points to help describe locations and movements. Grids provide a frame of reference for the location and movement of spatial objects and functions. We use grids to parse space on two planes, pinpoint locations and describe the relationship between objects, interpret motion, structure movement, and structure perception.

Grids as an underlying structure in mathematics classrooms

Grids are often used in mathematics classrooms, however they are applied implicitly for the most part, without instruction or even naming the underlying structure (Outhred & Mitchelmore 2000). Further, grids are rarely examined as a spatial object or tool in the mathematics classroom (Battista 1999, Outhred & Mitchelmore 2000). Students use graph paper, quick sketches or online applications (e.g., Mathigon, Desmos etc.) routinely. Typical textbooks and math teaching resources reveal that grids are commonly used to visually model a range of mathematical concepts including multiplicative relationships, fractions, measurement concepts, navigating locations etc. (see Figure 4). As an example, a teacher might use a grid structure to visually model the distributive property of multiplication over addition. When using grids in class, attention of educators and students is most often centered on the application at hand (in this example, visualizing a property of multiplication) while the spatial features of a grid's structure provides a backdrop (see Figure 4).

Figure 4.

Examples of commonly used classroom applications on grids



Note. From *Elementary and Middle School Mathematics : Teaching Developmentally* (Fifth Canadian edition, *p. 150, p. 270. p. 388*), Van de Walle, Bay-Williams, J. M., McGarvey, L. M., and Karp, K. S. (2016)., Pearson Canada.

In school mathematics, it is safe to say that the spatial structure of grids is rarely the focus of student explorations (Battista 1999, Outhred & Mitchelmore 2000). Students may not be attending to the properties of grids or transferring an understanding of this structure across concepts (see Section 2.1.7 Transferability). Although students encounter grids routinely, the fundamental properties of a grid's structure may not be apparent and have in fact been shown to be difficult for students to generalize (Barrett et al., 2017, Battista 1999, Battista et al., 1998, Battista & Clements 1996, Clements et al., 2017, Outhred & Mitchelmore, 1992, Outhred & Mitchelmore 2000, Sarama et al., 2003). Considering grids are a foundational spatial object underlying many significant mathematical concepts, a main objective of this study was to analyze how students perceive, use and learn about grid structures while engaged in a range of spatial tasks.

2.1.3 Student perceptions of grids

The spatial features of grid structures are not inherently obvious to all students (Outhred & Mitchelmore 2000). Researchers "suspect that many teachers perceive the structure of array as self-evident, without realizing the difficulties children face" (Outhred & Mitchelmore, 2000, p.147). One could argue that supporting students in "seeing" a grid's structure requires intentionally designing experiences with grids to help make its properties explicit. Despite well-established evidence on the importance of developing students' spatial thinking "current curricula and overall mathematics instruction rarely focus on spatial reasoning as a curriculum goal" (Bruce et al, 2015 p. 85). Considering the gap in research addressing how students perceive, use and learn about grid structures, combined with the buried nature of the spatial elements within the current curriculum, it follows that there is likely little intentional instruction given to this spatial object. My observations and experiences have borne this out in Ontario classrooms.

As such, it is not surprising that there is much we do not know about how to support students' perceptions of grid structures. As Julie Sarama stated, "the development of these ideas in the elementary years has not been adequately studied" (2003, p. 278). Later, Clements et al., (2017) echoed the deficit situation and remarked; "we were acutely aware of the limitations of the research base" during an investigation into how children perceive the spatial relationships within grids to support measurement concepts (p. 75). As stated in the previous section (see Section 2.1.2 Grids in the mathematics classroom) grids are often in the background of the application students are engaged in, likewise grids also underly much of the mathematics examined in educational research (e.g., measurement, coding, location and movement, multiplicative thinking etc.) but are rarely the object of study.

As an exception, in a study of 45 children in Grade 3 and 78 children in Grade 5, Battista & Clements (1996) documented the struggles students experienced in perceiving the spatial structure of 3D cubes and questioned whether this was a result of students' difficulty coordinating the row and column structure within the 2D layers. Two years later, in 1998, Battista et al., interviewed twelve second grade students in their investigation into how students' structure 2D space. There are three significant outcomes to this 1998 study in the research of grid structures. First, the authors uncovered a wide range of student understanding and used these findings to generate a continuum of student perceptions of grid structures (see Section 2.1.4 Developmental progressions). Secondly, as a result of these investigations Battista established a working definition for spatial structuring that is widely used today: “spatial structuring is the mental operation of constructing an organization or form for an object or set of objects" (Battista, 1999, p. 171). Finally, the third result of the Battista et al., 1998 study was the development of assessment tasks that could be used to uncover students' conceptions of grids. The seventeen tasks developed by the researchers all followed a similar structure. Each task required students to look at different grids that had elements removed. The seventeen different grids that students were shown varied in size and shape. The interior lines were removed to varying degrees; some displayed only one row and column, others were completely empty except for tick marks showing where the lines would have been. Students were asked to estimate how many square tiles would cover the space. They were then given a pencil and asked to ‘complete the grid’. Several studies, including this thesis, have used variations of the tasks developed by Battista et al., 1998 to examine students' spatial structuring of grids (Barrett et al., 2017, Clements et al., 2017, Mulligan & Mitchelmore 2009, Mulligan et al., 2020, Outhred & Mitchelmore 2000,).

Developing student perceptions of grids

Another significant study in 2009 by Mulligan and Mitchelmore examined students' spatial conceptions of grids. This study interviewed 103 first grade students (aged 5-6), prompting them with a series of 39 tasks designed to examine a new construct that the authors describe as students’ Awareness of Mathematical Pattern and Structure (AMPS). One of the tasks the authors used in this study was an adaptation of the Battista et al., 1998 task where students saw partial grids and were asked to estimate their area and complete them through drawing. The Mulligan and Mitchelmore

(2009) study reports success in helping students make generalizations about grid structures by drawing students' attention to the discrepancies between grids they "see" in their minds eye and the grids they ultimately draw. Battista (1999) also named this strategy of drawing students' attention to comparing their perceptions of grids and their own drawing of grids as key to developing their spatial structuring of grids; "to construct a proper spatial structuring of two-dimensional arrays of squares, students need numerous opportunities to structure such arrays and to reflect on the appropriateness of their structurings" (p. 174).

In the 2009 study, Mulligan and Mitchelmore reported that encouraging students to focus on "spatial patterns leads to concepts such as collinearity, congruence and symmetry and the formulation of general properties of basic two-dimensional figures" (p. 42). These findings were confirmed through a much larger follow-up study in 2020 (Mulligan et al., 2020). In this 2020 study, the researchers were able to follow 319 Kindergarten students over two years as teachers implemented their Pattern and Structure Mathematics Awareness Program (PASMAP, formerly known as AMPS). As part of the PASMAP students were again shown partially constructed grids and asked to fill in the missing elements with a focus on attending to the repeated patterned elements of grids (Mulligan et al., 2020). Specifically, students were asked to attend to the pattern of squares; "students investigated cut-outs of grid cards in rows or columns with the number of small squares resulting in the pattern (1, 4, 9, 16, 25). Students were asked to make and draw the pattern from memory: "What comes next in the pattern so it is getting bigger each time?" (Mulligan et al., 2020, p. 672). The results of this 2020 study echo the results from the researchers' 2009 study where students increased spatial structuring of grids was reported as “critical to the students’ mathematical development reflected in concepts such as partitioning, congruence, equal spacing and co-linearity” (Mulligan et al., 2020, p. 674).

The Australian researchers acknowledge that their investigation into spatial structures through the Awareness of Mathematical Pattern and Structure (AMPS) project is unique as “there have been remarkably few studies that have attempted to describe general characteristics of structural development in young students’ mathematics” (Mulligan & Mitchelmore, 2009, p. 29). While greater interest and attention has been given to spatial research and the corresponding implications for instruction in recent years, notably in Canada through the Math 4 Young Children (M4YC) initiative,(www.tmerc.ca, see also Moss et al., 2016), investigation into how students’ learn about and make use of grids as spatial objects remains limited, yet rich with potential.

2.1.4 Developmental progressions (application as mathematical end goal)

The research into students’ spatial structuring of grids has largely focused on generating developmental progressions (see Table 1 below). These developmental continua help to distinguish changes in students’ reasoning and interactions with grids as they transition toward more sophisticated understanding. The sequence of missing grid line drawing tasks to assess conceptions of area, designed by Battista et al., (1998) has provided a common starting point that researchers have used and adapted to help uncover how students “see” grids. Broadly speaking the research suggests that “students’ understandings of array structure progress from a collection of individual units to (perpendicular) intersecting sets of parallel lines” (Outhred & Mitchelmore, 2004, p.

465). To further summarize the literature to date on developmental continuums on student conceptions of grids, the following table (Table 1) aligns four seminal research papers in terms of the developmental levels identified in mathematics education research. These continua will be subsequently discussed from early to mid to late stages of understanding as conceived and described by these same researchers.

Table 1.

Alignment of developmental continuum of student conceptions of grids

Battista et al., (1998)	Outhred & Mitchelmore (2004)	Mulligan & Mitchelmore (2009)	Clements et al., (2017)
	Level 0: Incomplete Covering	Prestructural	Area Quantity Recognizer Physical Coverer and Counter
Level 1: Complete Lack of Row Column Structuring	Level 1: Primitive Covering	Emergent	Complete Coverer and Counter
			Area Unit Relater and Repeater
Level 2: Partial Row Column Structuring	Level 2: Array Covering Constructed from Unit	Partial Structural	Initial Composite Structurer
Level 3A: Structuring an Array as a set of Row or Column Composites	Level 3: Array Covering Constructed from Measurement		Area Row Column Structurer
Level 3B: Visual Row or Column Iteration			

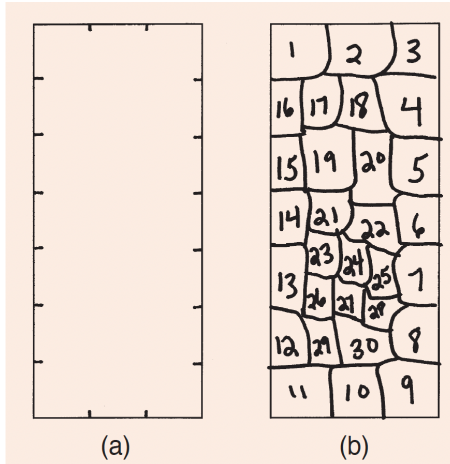
Level 3C: Row by Column Structuring	Level 4: Array Implied Solution by Calculation	Structural	Array Structurer
		Advanced	Conceptual Area Measurer

Early conceptions of grids

In the earliest stages of these continua, students are not able to perceive any larger spatial structure and instead “the grid itself may be viewed as a collection of square regions” (Sarama et al., 2003, p. 288). Sarama and Clements, two American researchers who have been studying mathematics learning for decades, have identified a range of foundational mathematics learning trajectories that support incremental student understanding of key ideas. As it relates to grid structure, they have identified challenges students face with making sense of intersecting regions. They note that difficulty “seeing” the structure often results in students counting cells “around the border, spiraling to the center” cell by cell (Clements et al., 2017, p. 73). They may heavily rely on tick marks and other spatial unit indicators in order to engage with the structure (Cullen & Barrett, 2020, Clements et al., 2017). A common error students make is to at first use the support of grid lines as they spiral in their count around the perimeter of the area they are counting. As their count moves towards a less discernible centre the space becomes undefined and it is common to see double tapping, blurred counting or drawing of non-uniform cells (Battista et al., 1998, Battista 1999, Clements et al., 2017) (see Figure 5).

Figure 5.

The spiral-like counting pathway of a second-grade student attempting to construct a grid



Note. From “The Importance of Spatial Structuring in Geometric Reasoning” M. Battista, 1999, Teaching Children Mathematics, 6(3), p. 170 (https://doi.org/10.5951/TCM.6.3.0170)

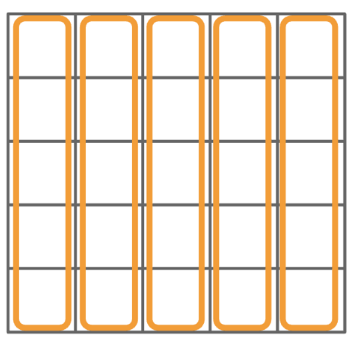
The individual cell drawings of varying size, along with the spiral shape of the count pathways as described by Battista et al. (1998), Barrett & Clements (2017), Mulligan et al., (2020) and Outhred & Mitchelmore (2004), point to students at the early stage of these continua as being unable to coordinate the two-dimensional nature of grids. In fact researchers note that it is difficult early on to determine if students are able to consider two-dimensions; “knowing whether children are conceptualizing two-dimensional shapes (congruence or area) or just the one-dimensional side (length) is difficult” as children may instead be “comparing shapes by using this side-matching strategy” (Clements et al., 2017, p. 74). Overall, research on conceptions of grids for the youngest learners indicates that students are likely perceiving an unsystematically arranged collection of squares (Battista et al., 1998, Barrett et al., 2017, Clements et al., 2017, Outhred & Mitchelmore 2000).

Partial and emerging conceptions of grids

As students' spatial structuring evolves, they begin to “see” collections of individual units as part of larger composite units within grids. A composite unit is “formed when a student takes a set of unit items and treats it spatially or numerically as a unit” (Battista, 1999, p. 173) (see Figure 6). As students structure rows or columns of composite units, they must generalize the equivalence of the rows/columns and apply this “uniform organization” across grids (Battista et al., 1998, p. 520). Students who recognize the spatial structure of rows or columns may start to draw straight lines across grid space as opposed to individual cells. At this stage students are only attending to “parts” of the larger array structure focusing on a local as opposed to a global structure (Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2009).

Figure 6.

Five individual cells become a composite unit of 1 column that can be iterated and counted as a unit (e.g., counting 1,2,3,4,5 groups of five, or a count that repeatedly adds groups of 5,10,15,20,25)

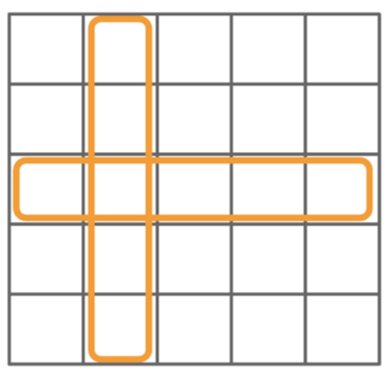


Advanced conceptions of grids

A common distinguishing feature of students at the far end of the developmental progression for spatial structuring of grids is the ability to coordinate dimensions. Students at this stage have “structured the array into two coordinated orthogonal dimensions" (Battista et al., 1998, p. 523). They are able to "see" that "the elements of an array are collinear in two directions" (Outhred & Mitchelmore, 2004, p. 468). The spatial organization and coordination they have applied to grids supports their generalization about the relationship between rows and columns (see Figure 7). By coordinating spatial row/column structures students are able to apply more sophisticated enumeration strategies to grids (Battista et al., 1998, Barrett et al., 2017, Clements et al., 2017, Cullen & Barrett, 2020). Students at this developmental stage can apply a global structure considering grids as a whole structure (Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2009, Outhred & Mitchelmore 2000). Coordinating parts of grids within the overall grid structure involves a cognitive process known as simultaneous processing (Mannamaa et al., 2012) (see Section 2.1.12 Coordinating dimensions).

Figure 7.

For every one column of five there is a corresponding row of five, the space and quantity are organized into a five by five array



2.1.5 Grids as a series of regions

As noted previously, much of the research on how students' structure 2D space is situated within investigations into students' development of area measurement concepts. It is through these measurement investigations of children ages 4-13, that we have come to understand how complex and diverse students' spatial structuring of a given area can be (Barrett et al., 2017, Battista et al., 1998, Clements et al., 2017, Outhred & Mitchelmore, 2000, Mulligan et al., 2020). The challenging nature of conceptualizing two-dimensional space is contrasted by the extensive use of the formula, area = length x width, which students often apply "only as a rote procedure, without understanding its mathematical foundation" (Clements et al., 2017, p.74). Researchers have long understood that when this formula is applied without conceptual understanding "it is likely that for some of these students, "square units" do not conjure up an image of a square" (Simon & Blume, 1994, p. 485). Research findings into the complexity of how students conceptualize two-dimensional space have largely been revealed through the analysis of student drawings of arrays and students' corresponding attempts to numerate the space as they "see" it (Battista et al., 1998, Barrett et al., 2017, Clements et al., 2017). This is important because how students perceive a grid's structure impacts how they enumerate the space (Barrett et al., 2017, Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2009). Battista et al., (1998) found that "spatial structuring preceded *meaningful* enumeration" (p. 504). Along the noted developmental continua researchers described the numeration of early conceptions of grids as expressed through a count of individual cells, whereas mid-level conceptions began to group individual cells into units (along either rows or columns) and finally more advanced conceptions employed multiplication of rows and columns (Barrett et al., 2017, Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2009).

The spatial structuring of area contributes to part of a collection of processes, including covering, quantifying and subdividing, that together contribute to a students overall development of measurement understanding (Clements et al., 2017). Within a measurement context, spatial structuring "involves recognizing the goal of partitioning a region into parts" as well as "the activity of organizing the region into a row-and-column structure" which results in "a fully partitioned and quantifiable region" (Clements et al., 2017, p.74). Spatial structuring requires: (i) the simultaneous processing between the parts of a given region; (ii) the arrangement of these parts within a row and column structure; and, (iii) ultimately the coordination of these parts within the larger structure of the overall area.

As students structure the space towards the goal of quantifying the area their attention is directed towards the regions within the rows and columns on grids or partitioned areas. It is the regional space (area) created by the partitioning grid lines that is being quantified. This is not inherently obvious - "for some students the lines shown in an array may be only a visual feature unrelated to numerical structure" (Outhred & Mitchelmore, 2004, p. 471). Over time, with intentionally structured experiences, attending to grid structure, students spatial structuring can be developed to support their numeration from early counting in "a disorganized manner" to later using the row and column structure to support strategies for performing multiplication or repeated addition (Outhred & Mitchelmore, 2004, p. 465).

Conversely, students who have developed a more sophisticated ability to spatially structure areas are described as being able to restructure rows and columns to coordinate with given measurements (Clements et al., 2017). Students must simultaneously attend to the construction of equally sized individual regions as well as creating equal partitions along the length and width dimensions to fit specified measurements. Coordinating the spatial structure of these regions requires “a good understanding of linear measurement, without which children are unlikely to learn the relation between unit size and rectangle dimensions” (Outhred & Mitchelmore, 2000, p. 165). This harkens back to distance conceptions of grid structures outlined in the thought experiment at the opening of the literature review.

It is important to recognize that the field of research on how students think about area measurement concepts is somewhat narrow; “overall, only a limited number of studies discussed children’s learning of area” (Clements et al., 2017, p. 75). Within this small collection of studies, the spatial structuring of 2D space is part of a broader investigation into students' development of measurement concepts. It is clear that the vast majority of research into how students construct and make use of the spatial structure of grids is largely bound within a measurement context and as such often describes students' construction of grids as a series of regions. This is extremely important work, but also requires expansion well beyond area conceptions of grids.

2.1.6 Grid as a coordinated relational object

When engaging in coordinate mapping, location and movement situations students must reorient their perception of grid structures from that of a series of regions to that of a coordinated system of lines that organize and structure relationships between objects. This is an important distinction which is sometimes overlooked (Sarama et al., 2003). It can be quite challenging for students to “utilize a conceptual coordinate system as an organizing spatial framework” (Sarama et al., 2003, p. 287). Researchers have observed that one aspect of this difficulty stems from the discrepancy between viewing grids as “a collection of square regions, rather than as sets of perpendicular lines” (Sarama et al., 2003, p. 288). Students must attend to grid lines and the intersection of these lines as the emphasis shifts towards locating and coordinating space as opposed to quantifying space. The shift in application and context corresponds with a shift in what constitutes a “unit” (from one individual region to one length of that region vertex to vertex). Researchers noted that confusion over what features of grids should be attended to, and likewise students conceptions of what was being quantified (one unit as a cell or as a length) required intentional instruction (Sarama et al., 2003). In some ways, this involves a shift from cells of grids as iterations to cells of grids as intervals in equidistant and intersecting spaces.

Sarama et al., (2003) suggested that “students may have to finish building a horizontal conceptual ruler as a mental object on grids before reconstructing one in a vertical direction and then, finally, coordinating and synthesizing the two to reconceptualize space as being segmented by two orthogonal number lines” (p. 308). Students likely require additional support to develop their fluency with structuring vertical space as their regular classroom experiences lean towards partitioning space on a horizontal number line or ruler. This favouring of the horizontal number line corresponds with the predominant Western conception of “numbers as ordered magnitudes along a left-to-right axis” (Hawes & Ansari, 2020, p. 469). Researchers also observed that when partitioning regions on grids “children almost always constructed and counted their arrays by rows and not by columns” likewise favouring the horizontal space (Outhred & Mitchelmore 2000, p. 154). (see Section 2.1.11 on Linear representations)

2.1.7 Transferability

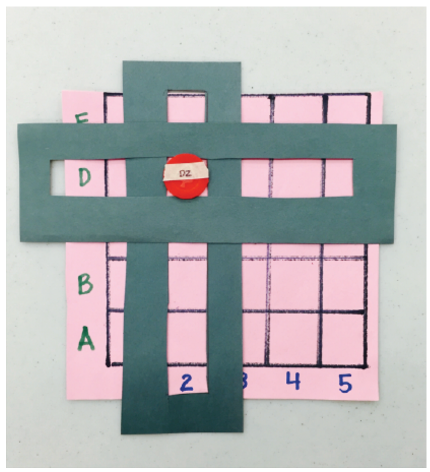
Researchers acknowledge that student abilities to perceive grid structures as “representing an array of units using two perpendicular sets of parallel lines is more difficult than might be expected, indicating that the structure of a square tessellation is not obvious to students but must be learned” (Outhred & Mitchelmore, 2004, p. 470). When students do learn about grid structures it is often indirectly (e.g., through the exploration of a measurement task). Which aspects of grid structures students attend to is dependent on the given classroom application. In one part of the curriculum, the teacher might draw students’ attention to the area regions within grids while exploring multiplication. Later the teacher may direct students to attend to the intersection of lines during a mapping, location and movement focus. While students are likely engaging with grids across various areas of study in the mathematics classroom (and possibly in other STEM disciplines), I hypothesize based on my experiences in mathematics education and in the literature review conducted, that it is unlikely that the spatial structure of grids has been made explicit as students transfer between different types of uses. More often, students experience grids in isolated situations or in the background of the application at hand. As we transition from one emphasis to another (e.g., thinking of grids as regions in area applications to thinking of grids as intersecting lines in location and movement applications) how do students transfer and apply their understanding of the underlying grid structure? Researchers have observed that students who are successful in adapting and transitioning between applications on grids “realize that the structure is more general than the specific task, enabling them to succeed on similar tasks or simple extensions” (Mulligan & Mitchelmore, 2009, p. 41). Mulligan and Mitchelmore note that many students do not seem to transfer their understanding of spatial structures between applications and require intentional instruction to support students in constructing generalizations about these objects (Mulligan & Mitchelmore, 2009). Ultimately the goal would not be to didactically teach students the spatial features of grids, instead we might ask what kinds of experiences allow students to notice and play with grid structures as they construct meaning for themselves.

Coding and grids use

Research into early unplugged coding experiences for students reported students' spatial structuring of grids as having the “potential to unlock foundational constructs related to coding” (Flynn 2018, p. 169). Flynn and Bruce’s classroom-embedded study engaged young children in playful unplugged coding experiences and developed potential tools for highlighting important features of grids. As an example, when noticing young children struggling to isolate intersecting spaces on grids; “the team did more to support student understanding of the grid and its structure (for example, the “window finder” activity which helped to highlight the row and column array structure of the grid and to isolate intersections/locations)” (see Figure 8) (Flynn 2018, p. 167). This is an example of a classroom educator and researcher co-development of a support for making grid structures explicit to young children (ages 4 and 5) within a playful and engaging context. There is potential for these same “window finders” to support students in isolating rows and columns on grids as well as ‘seeing’ intersections between rows and columns. Applications in classroom contexts beyond coding could also support students in transferring their perceptions of grids across contexts, for games, mapping and locating and understanding the concept of intersecting spaces.

Figure 8.

Example of “window finders” developed to support students in locating intersections of rows and columns on grids



From. Mapping a learning trajectory and student outcomes in unplugged coding: a mixed methods study on young children’s mathematics and spatial reasoning. T. Flynn, 2018, Trent University. Educational Studies M.Ed. Graduate Program, p. 119.

2.1.8 What is a spatial object?

In this literature review, I have discussed the existing literature on the developmental continua of student understanding of grids, particularly in the contexts of measurement. However, there is much more to explore when we consider grids as spatial objects that are unbound by typical curriculum compartments. What exactly is a spatial object? And why is this conceptual framing important to understanding grids with young children? Newcombe and Shipley (2015) provide an expansive definition of the term 'object';

> Specifically, what we call objects will vary at different spatial scales. For example, an astrophysicist may treat a galaxy as an object, whereas a microbiologist may treat a cell as an object. For any given spatial task, the appropriate scale is determined by specifying what constitutes an object (a molecule, a chair, a house, a country, a planet, a solar system, etc.). An object can, in principle, be defined at any scale, although some objects are privileged by virtue of being the entities that humans naturally manipulate in their everyday lives. (p.4)

A spatial object is therefore defined and determined by the observer and the context with which they are observing the object. Grids are spatial objects in and of themselves, they can be tangible physical objects (graph paper, lines drawn through the sand etc.) or imagined objects that serve as mental constructions allowing us to conceptualize mathematical metaphors such as continuous and infinite space, parallel lines and infinite points along those lines (Lakoff & Nuñez, 2000). The spatial structure of grids ultimately brings "together arithmetic, analysis, geometry, and logic" as they link "number and shape through a coordinate system" (Davis et al., 2015, p. 50). Grids are spatial objects that instantiate number and shape simultaneously (Davis et al., 2015).

However, the spatiality of grids may not be perceived as important to the observer, depending on the context. Grids may be background to the spatial context the observer is studying. For example, a grid may simply serve as a partitioned plane on which another spatial object, such as a shape, is being considered. Imagine the rotation of a right angle triangle on a desktop. Now imagine that same triangle rotating on a grid. The grid can helpfully operate to make sense of the space, the movement of the triangle, its orientation and its location. At other times grids may be foregrounded, but the focus is on the application, such as when we are comparing the area of two irregular figures drawn onto a grid. In this case, grids are being used as a tool supporting the application and less so as a spatial object being examined by the observer.

There are occasions however where grids may be observed and defined as a spatial object. For example, the observer may explore the relationships between the intersection of perpendicular lines, the movement of angles and dimensions and the corresponding impact across grids (Sarama et al., 2003, Mulligan et al., 2020). In this case grids are the spatial object being examined, in and of themselves. Given the pervasiveness of grids throughout a wide range of mathematical contexts, whether it is foregrounded or backgrounded, it follows that careful study of grids as a spatial object could support researchers, educators and students alike in exploring spatial reasoning across the spectrum of mathematics learning (Battista 1999, Outhred & Mitchelmore 2000, Mulligan et al., 2020).

2.1.9 Considering grids within a typology for spatial thinking

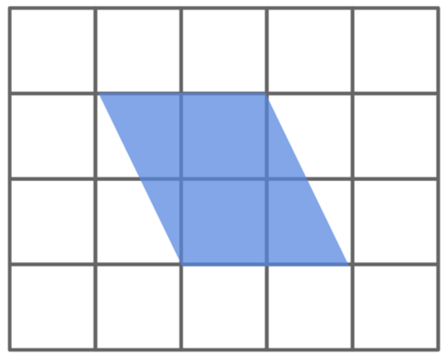
Newcombe and Shipley (2015) proposed a typology for thinking about spatial thinking. They distinguish between intrinsic understandings (the parts or internal structure of an object) and extrinsic understandings (the relationship of an object to other objects and/or the environment) understandings (Newcombe & Shipley, 2015). The intrinsic or extrinsic spatial information of an object can also be viewed as static or dynamic in nature. Generally speaking, “static skills involve coding of spatial object features, locations, and configurations” whereas “dynamic skills involve the transformation of these spatial codings, locations, and interrelations” (Frick, 2019, p. 1468). Newcombe and Shipley (2015) cross reference the intrinsic and extrinsic types according to either static or dynamic elements ultimately creating four distinct categories; intrinsic-static, intrinsic-dynamic, extrinsic-static and extrinsic-dynamic (IS, ID, ES, ED) for analyzing spatial thinking (Newcombe & Shipley, 2015). As spatial objects, the prevalence and range of utility of grid structures can be viewed across each of the four IS, ID, ES and ED categories.

Intrinsic-static

One element of the intrinsic-static quadrant involves examining the internal structure of objects and “identifying regions of space as constituting categories” (Newcombe & Shipley, 2015, p. 5). This process involves the spatial skill known as disembedding (Newcombe & Shipley, 2015). Disembedding the spatial information within grid structure itself involves “seeing” rows and columns, individual cells as areas and the tessellation of squares. Whereas disembedding the spatial information of figures on a grid involves using a grid’s structure as a reference to support isolating the spatial relationships within the object. (see Figure 9)

Figure 9.

Grids supports “seeing” shapes within a shape, we can disembed triangles from within this parallelogram

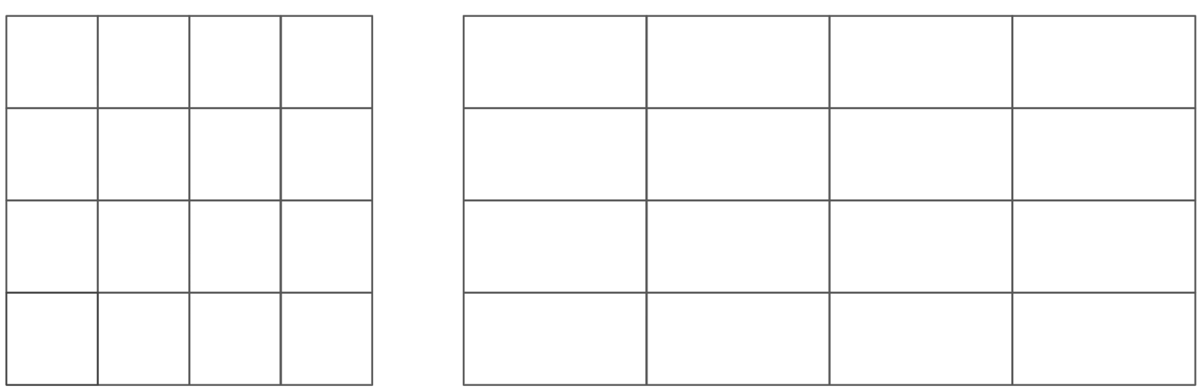


Intrinsic-dynamic

When considering grids as spatial objects, from an intrinsic-dynamic perspective, the internal spatial information of grid cells become malleable such as when we stretch the height or width of cells, or stretch points diagonally. Since traditional square grid structures are traditionally static in nature, pursuing dynamic girds provides an opportunity to analyze its internal structure and the impact of change on that structure. (see Figure 10).

Figure 10.

As the width of each column increases so does the area within each cell



We can also use grids to support the kinds of spatial visualizations needed to think about the intrinsic-dynamic nature of an object. For example, cross-sectioning falls under the intrinsic-dynamic category as it “refers to the ability of visualizing changes to the internal structure or shape of an object” (Frick, 2019, p. 1468). Grids may be used to support imagining the partitioning of a figure providing new insight into the spatial configuration of the cross section (e.g., visualizing the cross section on a grid, visualizing a grid wrapping around an object being sectioned etc.). Cross sections supported by the spatial structure of grids have many applications from geology to design to gaming and computer animations.

Another critical element of the intrinsic-dynamic subset is the ability to spatially rotate objects; “the ability to mentally rotate objects in space has been singled out by cognitive scientists as a central metric of spatial reasoning” (Bruce & Hawes, 2015, p. 331). Most often the tasks used to assess students' mental rotation “require the participants to recognize spatial symmetry (Frick, 2019, p. 1479). Grid structures may be useful in providing a frame of reference to support the mental rotation of objects and to recognize symmetry.

Extrinsic-static

Perhaps most notable is the prevalence of grid structures within the extrinsic-static domain. This branch of spatial thinking involves “coding the spatial location (or position) of objects relative to other objects or to a reference frame, including gravity; aligning relative to other objects or to a reference frame” (Newcombe & Shipley, 2015, p.5). A “reference frame”, as noted by Newcombe and Shipley (2015), could refer to an organized, potentially labelled space, most often represented by a grid. Grids allow us to see, describe and relate objects to each other and to their environment by the nature of their spatial composition. The consistent measured space supports the gauging of distance between objects, between objects and the environment as well as the position or orientation of objects in relation to each other. Even in the absence of a grid we can imagine or project a grid structure onto space to use as a reference frame. The definition of extrinsic-static skills provided by Newcombe and Shipley (2015) describes how integral being able to imagine the organization and structure of space is to this category; “extrinsic-static skill (which has been called spatial perception but which is essentially the ability to accurately code horizontal and vertical dimensions as defined by gravity)” (Newcombe & Shipley, 2015, p.5).

Extrinsic-dynamic

Lastly, extrinsic-dynamic thinking involves “transforming the inter-relations of objects as one or more of them moves, including the viewer (e.g., to maintain a stable representation of the world during navigation and to enable perspective taking)” (Newcombe & Shipley, 2015, p.5). Navigation and perspective taking are fairly broad and widely researched areas of spatial reasoning (Frick et al., 2014). When assessing students, researchers often use “the classic perspective taking task [which] involves arrays of multiple objects and tests with picture choice and model building” (Frick & Newcombe, 2014, p. 6). These tasks intentionally do not include an external frame of reference, such as a grid structure in order to avoid complicating the analysis (Frick & Newcombe, 2014).

The question remains: If students build perspective taking skills through the support of grid structures during learning opportunities, would they be better able to structure space in the absence of grid structures during an assessment? Grid structures can be applied to all four categories (intrinsic-static, intrinsic-dynamic, extrinsic-static, extrinsic-dynamic) and in this way support the argument that grids as spatial objects are foundational within mathematics education. Depending on the context it may be integral to making sense of the spatial information (such as mapping and locating, measuring) or it may be supplementary (such as provide a frame of reference for mental rotations or defining space when looking at cross-sections). Mapping out how grid structures support spatial reasoning across these four categories identified by Newcombe and Shipley could provide greater insight into the prevalence and importance of grids as spatial objects.

2.1.10 Dynamic and mutable grids

How we map space and number most certainly predates Descartes, however “the natural relationship between algebra and geometry, and the Cartesian coordinate system became an extremely convenient “handle” by which to refer to positions in space” (Kim, 2001, p. 242). A well-known story that accompanies Descartes’ formalization of a coordinate system recounts a young sickly Descartes spending his days staring at the ceiling trying to describe the movement of a fly as it moved around above him (Kim, 2001). Descartes was developing a way to describe movement in space in three dimensions. Descartes' development of the coordinate system was also rooted in his fascination with solving the classic Pappus’s problem which involved defining properties of curves (Gowers, 2008). While visually working through Pappus’s problem, Descartes “introduced coordinates *x* and *y,* using oblique as well as rectangular coordinate axes, which he always adjusted to the problem at hand”

(Gowers, 2008, p.739). It is interesting to consider the origins of the now widely used Cartesian coordinate system as these stories describe imagining adjustable grid structures to support thinking about movement and shape. The dynamic and mutable nature of grids is inherent in its origins, however, student experiences with grids are largely in a static fixed state; “this is evident in the Cartesian coordinate system, which, despite the fact that they are meant to describe movement (speed and acceleration) are often interpreted by teachers and students as static objects” (McGarvey et al., 2015, p.109).

The static nature of grids aligns with what researchers have reported as common practice in elementary mathematics education;

> This simplified and static conception of geometry is reflected in current mathematics programs in Canada and the US—where objects are rarely moved, transformed or re-shaped, and figures in the environment are rarely visualized from different perspectives. Although neglected, dynamic transformational geometry—thinking about how shapes move, change, interact in space, and how we move in relation to shapes and figures—is an important construct of geometry, namely that of spatial reasoning. (Bruce & Hawes, 2015, p. 331)

This literature review has repeatedly demonstrated that young children struggle to generalize the spatial structures of square grids (Barrett et al., 2017, Battista et al., 1998, Battista 1999, Clements et al., 2017, Mulligan & Mitchelmore 2009, Mulligan et al., 2020, Outhred & Mitchelmore 1992, Outhred & Mitchelmore 2000, Sarama et al., 2003). Perhaps in its fixed static state the spatial structure of a square grid become less obvious to the observer. Researchers are noticing that it is through the active building and construction of grids that students are able to generalize its structure (Battista 1999, Mulligan & Mitchelmore 2009, Mulligan et al., 2020). Movement seems to have an impact on the way we take in and process spatial information (Holmes et al., 2018). In a study of 125 undergraduate students, researchers asked participants to observe an array of objects laid out on a table. Some students had to remain fixed when observing the objects, while others could walk around the objects and examine them from multiple perspectives. Holmes, Newcombe and Shipley reported that the "participants who moved around the array formed more accurate and flexible spatial memories" (2018, p. 23). There is potential for other active experiences such as bending, stretching, scaling and twisting to help the spatial features of grids come alive for students.

2.1.11 Coordinating Dimensions

Coordinating the spatial structure of a square grid requires the viewer to oscillate between "seeing" details of specific parts (a cell or an intersection) and groups of parts (such as a row or column) as well as "seeing" the overall structure (grid as a whole object). Here, I propose that the coordination between the spatial information of the parts of grids as well the spatial information of the whole grid involves simultaneous processing. Simultaneous processing is a concept introduced by Mannamaa (2012) which involves "making perceptual gestalts or spatial groups" which allow the viewer to perceive the whole at once (2012, p. 36). We can think of simultaneous processing as supporting "visualization, whether it is seeing a picture or positions, orientation, shapes, or shapes in motion" (Tepylo, 2017, p. 108). Interestingly, simultaneous processing has been shown to support problem solving skills in mathematics as well as reading comprehension (Mannamaa et al., 2012). Problem solving skills require students to simultaneously coordinate specific information from within the problem to the broader context while evaluating strategies and working towards a solution (Mannamaa et al., 2012). Making sense of grids involves simultaneous processing as it involves visually isolating cells or rows and columns while also integrating these isolated parts within the larger overall grid structure.

As a result of their longitudinal work with young children through the Pattern and Structure project, referred to previously and beginning in 2001, Mulligan and Mitchelmore hypothesize that students' coordination of cells and row/column structures supported their generalizations across other areas of mathematics (e.g., pattern recognition, multiplication) (2009). Mulligan and Mitchelmore describe mathematical structure as “most often expressed in the form of a generalisation of a numerical, spatial or logical relationship which is always true in a certain domain” (2013 p. 34). The Australian researchers suspect that the way students recognize the spatial and numerical patterns within row and column structures of square grids supported their generalizations about multiplication (Mulligan & Mitchelmore 2013, Mulligan et al., 2020). Recognizing and isolating the “parts” (cells, rows, columns) within the larger structure of grids in order to generalize the overall spatial pattern and structure may be connected to students' ability to simultaneously process information.

2.1.12 Linear Representations

Thinking about numbers as continuous points on a line is “one of the most central concepts in all of mathematics” (Lakoff & Nuñez, 2000, p. 6). Number lines, like grids, are important spatial objects that organize and structure our combined sense of number and space. Researchers believe that our mental number lines “play an important role in performing a host of numerical reasoning tasks, including comparing, ordering, and operating on numbers” (Hawes & Ansari, 2020, p. 476). The mental number lines we imagine can be developed and refined with training, and the accuracy of their construction has been shown to be predictive of mathematics achievement (Booth & Siegler, 2008, Fischer et al., 2011, Siegler & Booth, 2004, Siegler & Ramani, 2009). The relationship between spatial ability, number line accuracy and math achievement are so intricately linked that in a longitudinal study Gunderson et al., was able to report that; “children’s spatial skill at age 5 predicted their number line knowledge at age 6, which in turn predicted their performance on an approximate symbolic calculation task at age 8 (controlling for vocabulary knowledge)” (2012, p. 1238). This is a critical study as it links students' spatial skills to their number line accuracy and general mathematical ability.

How we might support students in developing and refining their mental number lines is of great interest to educational researchers. What appears to be challenging for students as they develop their conception of the number line as a spatial object is the disconnect between the implied infinite space attributed to a line of numbers with the discrete quantities and bounded number lines students experience in classrooms (Newcombe & Shipley, 2018). Researchers speculate that children have “difficulties in connecting the discrete extensive information provided by counting (i.e., the number), which is prioritized at the beginning of school, with a continuous intensive conceptualization of quantity (i.e., the space between both ends of the number line)” (Newcombe et al., 2018). Newcombe and Frick hypothesize that our mental number lines go beyond our understanding of the integer system; they are the act of mapping this system onto an imagined continuous linear space (2018). Grids likewise present a challenge of mapping integer systems onto a two-dimensional construction of continuous space (see Section 2.2.2 Grids as objects-to-think-with).

There is a vast body of research into how humans construct their own internal mental number lines. One significant study found that people raised in Western cultures (who learned to read in a left-to-right direction) most often associate smaller numbers with being to the left and larger numbers to the right. This is commonly referred to as the SNARC effect (Spatial-Numerical Association of Response Codes) (Dehaene et al., 1993). The left to right bias (SNARC) is consistent with Susan Gerofsky's research into how grid structures are internalized. She described the left to right bias as an imagined *x*- axis stretching across the midline of the body with a corresponding *y*- axis down the centre line (Gerofsky 2010) (see Section 2.2.2 Grids as objects-to-think-with). In an Australian study of 115 grades 1-4 students, Outhred and Mitchelmore reported that students overwhelmingly favoured counting arrays by rows as opposed to columns (2000). Outhred and Mitchelmore's classroom embedded research aligns with a psychological study by Holmes and Lourenco (2012) who tested 52 undergraduate students and found the SNARC effect was stronger along the horizontal than vertical axis. Educational researchers hypothesize that students are perceiving grids as the coordination of intersecting number lines (Gerofsky 2010, Sarama et al., 2003). The way in which students are constructing and experiencing linear numerical spatial representations in school is potentially influencing their conceptions of grid structures.

2.2 Objects-to-think-with

The significance and utility of a robust imagination in mathematics cannot be overstated. This is particularly true in geometry; “there is a *compulsion* at the heart of geometry, a *must be* or a *cannot be* about the entire realm of *possible* configurations—possible, that is, in the imagination. For geometry constrains even our imaginations” (Sinclair et al., 2012, p. 26).

This section of the literature review will explore Seymour Papert’s description of objects-to-think-with as well as the possibilities of grids as objects-to-think-with.

2.2.1 Seymour Papert

In his influential work ‘‘*Mindstorms: Children, Computers and Powerful Ideas*’’ (1980), Seymour Papert recounts his personal experience growing up using gears to think about, imagine, play with and construct his understanding of mathematics. He describes his relationship to the gears as having an object-to-think-with that enabled him to access “many powerful "advanced" mathematical ideas” (Papert 1980, p. viii). An object-to-think-with is much more than simply a way to represent mathematical ideas. The way Papert describes his experiences using gears as objects-to-think-with was far more complex than simply a representation of mathematical ideas. Papert was thinking *with* the gears, using them to construct new understandings and to “see old ideas in a new way” (Bers, 2017, p. 1).

Papert fondly recalls his childhood playing with gears, observing and imagining the novel ways in which they operated, he notes the differential gear as particularly interesting as it “did not follow a simple linear chain of causality” (Papert 1980, p. vi). Papert describes how foundational his early play experiences were in providing him with an *object-to-think-with* that would help him engage with and construct mathematical concepts;

> Gears, serving as models, carried many otherwise abstract ideas into my head. I clearly remember two examples from school math. I saw the multiplication tables as gears, and my first brush with equations in two variables (e.g., $3x + 4y = 10$) immediately evoked the differential. (Papert 1980, p. vi)

Papert is describing how he was actively *using* the gears to think-with. The gears were not representing multiplication or functions, the gears were the way Papert was constructing and playing with mathematical ideas. Papert goes so far as to say that he wasn't just thinking about gears but there was also a connection with "'the body knowledge,' the sensorimotor schemata of the child" taking place, essentially "you can *be* the gear" (Papert 1980, p. viii). The gears were an object-to-think-with that Papert could call upon to shift his perspective; he could imagine himself as the gear and use that perspective to solve novel problems. This might lead us to ask: How could we invite students to play with and develop deep relationships with spatial objects so that they might be able to call them up voluntarily when needed to imagine themselves inside the problems they're solving?

2.2.2 Grids as objects-to-think-with

For Papert gears were an object-to-think-with that allowed him to consider a range of mathematical concepts. Less is known about how students could conceptualize grids as objects-to-think-with to potentially apply across a range of mathematical contexts. Up until now we have been considering grids as operating as a spatial object, as a terminal idea. Now let's consider what happens if grids become an object-to-think-with*,* an object a child could spontaneously select to use to make sense of a complex problem. Grids allow students to organize and partition space, track and describe movements and locations. Given how widespread grids are throughout the mathematics curriculum (e.g., measurement, proportions, functions, coordinates, fractions, multiplication/division, coding etc.) it follows that students would benefit from developing a robust relationship with this spatial object. The particular way in which grids invite students to structure space provides a system for thinking about many fundamental ideas in mathematics.

As an object-to-think-with the mutability of a grid's structure becomes evident as the learner controls and adapts its form to new situations. If students were given opportunities to construct their relationship to grids through experiences with dynamic grids there is the potential for them to relate to grids as objects-to-think-with in more dynamic ways (such as imagining grids wrapping around other objects or stretching or compressing grids to think about a particular problem).

This is a new area of growing research, a search in ERIC/Proquest for object-to-think-with and grid gave zero results. Removing the search for grid and looking just at the search term "object-to-think-with" provided twelve results. The majority of these results related to computer programming (referencing Papert's contributions to early coding through Logo). Let's examine two perspectives, that of Sarama et al., 2003 as well as the work of Susan Gerofsky (2004, 2010, 2011) as they offer some insights into what might be involved in the construction of grids as spatial objects-to-think-with.

A study by Sarama et al., 2003 involved four classrooms of grade four students, two used a unit designed by Clements et al. (1995) to engage students in location and area tasks involving grids, and two classes were used as a control. The tasks that the two classrooms engaged in were a mix of computer applications and paper tasks involving grids. Their study was part of a much larger research project concerned with field testing curriculum items. Sarama steps outside of the larger project to share some of the insights gathered related to students' spatial structuring of grids. As expected, the researchers found that students had difficulty with making sense of the spatial structure of the square grids. In particular they noted the coordination of two dimensions as uniquely challenging explaining that;

> this integration is a distributive coordination; that is, one conceptual ruler must be taken as a mental object for input to another, orthogonal, conceptual ruler. We hypothesize that this is possible due to the recursive characteristic of the human cognition; we operate on an experience, in this case a conceptual ruler (or set of conceptual rulers), with the same scheme that generated each of the elements of this experience, or mental object. (Sarama et al., 2003, p. 313).

Sarama is suggesting here, that there is something about the way students have constructed their conception of the number line that plays an important role in their conception of grids (2003). Sarama et al., (2003) describe overlapping vertical and horizontal “conceptual rulers” as informing students' conceptions of square grids as a mental object. This is an important insight into thinking about grids as objects-to-think-with as it suggests that students' relationship with the number line as an object-to-think-with is fundamentally related.

In similarly important work, Susan Gerofsky of the University of British Columbia, began a multi-year Graphs & Gestures project after noticing how she and her colleagues are using similar gestures when teaching functions (Gerofsky 2010). Her study involved 22 Grade 8 and 11 students as well as their teachers who were interviewed while gesturing the shape of the graphs of functions found in their calculus textbooks (Gerofsky 2010). Gerofsky describes students' internal conceptions of grids as related to the body; "the y-axis is generally taken as a metaphor for the vertical axis of the body or the spine; the x-axis is an extension of the middle horizontal line of the body or the waist" (2004, p. 146). She finds that how we imagine grids is a direct result of the way our physical bodies move through space;

> our bipedal, standing bodies have three axes, defining the three orthogonal dimensions: a left-right axis of bilateral symmetry, an up-down axis bounded by head and feet (and centered at the waist), and a front-back axis formed by our ventral and dorsal sides. Graphs on Cartesian coordinates use these three bodily axes respectively as the archetype to create a graphic grid with x-, y- and z- axes. (Gerofsky 2011, p. 5)

Gerofsky sees the mind and body as inseparable in students' conceptions of and experiences with grids (2011). This is important in thinking about grids as objects-to-think-with because it suggests that movement and gesturing are actions researchers could observe and promote when studying how children are using grids as objects-to-think-with.

2.3 Gesture, language and diagram as part of spatial reasoning

2.3.1 Why study gestures in mathematics education?

The hand and body movements that accompany speech, commonly known as gestures, are most often used when there is a need to convey spatial information (Alibali, 2005, p. 308). Gestural analysis is an expansive field of research. Notably, the foundational work of David McNeill (1992) provides a framework for classifying gestures into four categories namely; beat, deictic, iconic and metaphoric. These four categories are commonly used in gesture analysis research to typify gestures.

Gestures serve to “reflect mental images” of how the speaker interprets spatial information (Alibali, 2005, p. 308). Gestures are often employed when referencing abstract concepts, “particularly in areas such as mathematics” (Roth, 2001, p. 731). Careful examination of student hand gestures is perhaps most relevant in mathematics where the nature of the communication is in respect to abstract spatial concepts. Paying attention to the spatial information students convey through hand gestures provides unique insight into how students are structuring space when constructing their understanding of mathematics.

The research on the use of gesture is also robust in relation to mathematics education. Some of the key researchers of gesture are Canadian, including Radford and Sinclair – who have extended gesture research to the extensive field of embodied cognition. Embodiment is a particularly important consideration in mathematics, as de Freitas and Sinclair explain, “*concepts* are typically considered immaterial and inert abstractions acquired after a series of ‘concrete’ activities” (2013, p. 468). They argue “against this image of concepts as abstractions” and point to how we “come to know through the body (encountering)" as an area of mathematics education research that is at present, "poorly understood" (de Freitas & Sinclair 2013, p.468). Wolff-Michael Roth, an Australian researcher in embodied cognition, challenges educators and education researchers to reconsider what it means to understand learning, he says we need a radical approach if we are to investigate "how the world looks to students and what kinds of world they currently inhabit" (2010, p. 16). Similar to de Freitas and Sinclair, Roth advocates for an approach that positions all modes of expression (such as language, diagram and gesture) as in simultaneous relation; "writing and speaking are the two incarnated modes of language that are closely related and in fact interlaced with the hand" (Roth, 2010, p. 16).

If we are able to recognize the ways in which students are communicating their spatial reasoning through their gestures, we can use that understanding to provide more targeted feedback and instruction. Teachers can knowingly employ gestures that support the concepts they are trying to highlight with students (Ehrlich et al., 2006). One study looked at two groups of five-year-olds as they engaged in a commonly used mental transformation task, first developed by Levine et al., 1999. One group was asked to look at two images (shapes) and imagine the movement of those pieces through space coming together to form one whole image. The other group of students completed the same task except this time the two pieces were shown as physical objects. This physical shapes approach allowed the interviewer to demonstrate the transformational movement of the pieces, using a combination of physical tools, gesture and language to model the movement of the pieces coming together to form a whole. It was determined that the second group outperformed the first and the researchers' concluded that “using gesture to instruct children may have a profound and positive impact on the development of early spatial skills” (Ehrlich et al., 2006, p. 1267). The students were still performing mental transformations, but the demonstration with gesture illustrated an example of this transformation. The study also noted that children often communicate important information about how they are constructing their understanding of mathematical concepts through “gestures and not in their speech” (Ehrlich et al., 2006, p. 1266). We might therefore conclude that studying student gestures gives us access to more of the student thinking than is likely communicated through language alone.

Not only does the study of gestures in mathematics education research support researchers and educators in being able to “read” a student's understanding more fulsomely, it is also an under -used tool for supporting student learning; “when children are asked to instantiate a new concept in their hands, learning is more lasting than when they are asked to instantiate it in words alone” (Cook et al., 2008, pg. 1054). Teachers and students can express powerful spatial reasoning through their gestural communication. It is an element of classroom communication that can easily be overlooked considering that “using the body to represent ideas may be especially helpful in constructing and retaining new knowledge” (Cook et al., 2008, pg. 1054). Paying careful attention to student gestures could give us access to new ways of both understanding students' thinking as well as ways to construct new learning.

Studying gestures in mathematics education research is perhaps especially important when studying the spatial thinking of young children. While the gestures of young children are often examined by researchers “much less is known about how children's gestures develop and change" (Hostetter & Alibali, 2019, p. 738). Gesture analysis of students in the early years, a time of significant developmental change, could reveal a continuum of gestural markers aligned with shifts in conceptual understanding.

For the purposes of this thesis, gestures will not be categorized by type but instead will be investigated as an integrated part of students' communication to be considered alongside their language and drawings. This study is not attempting to conduct a formal analysis of students' gestures in and of themselves but rather sees the differences in students' gestures as connected to the differences in students' spatial reasoning which is the ultimate focus of this research.

2.3.2 Relating gesture, language and diagram

The relationship between gesture and language has been well documented, "decades of research have shown that gestures are intricately tied to language and thought (Hostetter & Alibali, 2019, p. 721). Research has revealed that students "frequently conveyed strategies in gesture that were not expressed in the accompanying speech" (Ehrlich et al., 2006, p. 1266). Goldwin-Meadows (2000) argues that "gesture has privileged access to information speakers know but do not express in words" (p. 231). Research has also revealed a phenomenon called gesture-speech mismatch where students' words and gestures don't necessarily align (Goldwin-Meadows, 2000, p. 232). During critical transition phases students' gestures may in fact precede their verbal communications providing researchers and educators with important assessment information of emerging understandings.

There is also a perspective in education research that looks at integrating the combined analysis of students' language, diagrams and gestures as part of a multimodal expressions of students' thinking that are deeply intertwined (de Freitas & Sinclair 2012, Chen & Herbst 2013, Ng & Sinclair 2015). Chen and Herbst (2013) advocate for analyzing students' language, gestures and diagrams simultaneously: They posit that “to effectively examine students’ reasoning through interactions with diagrams, both gestural and verbal expressions need to be observed” (Chen & Herbst, 2013, p. 286) because language, diagrams and gestures are interrelated instantiations of student thinking (Ng & Sinclair 2015, Chen & Herbst 2013). Considering a wider range of communication modes allows research to capture how students “develop and communicate complex explanations without the need to use formal mathematical language; the gestures may enable students to engage in arguments about geometric objects before all those objects have been conceptualized formally and represented in formal language” (Chen & Herbst, 2013, p. 286).

The key works that this thesis is referencing (Battista et al., 1998, Clements et al., 2017 and Outhred & Mitchelmore 2000) are examples of analyzing a combination of student drawings, gestures and language to capture a more fulsome portrait of student thinking. With the importance of this multi-modal approach to understanding student reasoning, this thesis aimed to analyse all three as a composite window into understanding how young children see and think about grids.

Methodology

3.1 Research design

This thesis represents exploratory research aimed at gathering a range of qualitative information through individual interviews with children (ages 4-9) as they engaged in playful tasks related to grids. Interviews were video and audio recorded to capture participants' gestures and language as they responded to questions about grids. The tasks were intended to be open and playful in nature to elicit students' intuitive and natural responses to square grids and their use in playful contexts. In line with Ginsburg's findings (2009), these task-based interviews were a valuable tool for collecting descriptive information about the children's mathematical thinking (Ginsburg, 2009).

Because the aim of this study was to uncover *how* young children conceive of the spatial features of square grids - essentially exploring the different *types* or ways students are engaging with grids - a qualitative interview-based research design was a good match for capturing the fulsome nature of young children's communication (i.e., through a combination of language, gesture and drawing).

One such qualitative research design that focuses on developing typologies, is known as 'ideal-type analysis', which allows for "rich description of what participants are saying, doing, thinking, and feeling, as well as providing an opportunity for interpretation, understanding, and explanation" (Stapley et al., 2021, p. 16). An ideal-type analysis is "a relatively new addition to the family of qualitative research methods, which offers a systematic, rigorous method for constructing typologies from qualitative data" (Stapley et al., 2022 p. 1). Following multiple stages of analysis, this thesis research involved developing a working typology through an ideal-type analysis of participant interviews. According to Mandara a typology is a “hierarchal system for organizing categories” which allows us to understand similarities and differences between groups (2003, p.132). The typology generated in this thesis mapped different categories of young childrens’ spatial structuring of square grids resulting from an ideal-type analysis of qualitative interviews (see Findings). Stapley et al., (2022) outline a step-by-step data analysis procedure for applying an ideal-type analysis which this thesis has applied as part of its design (see section 3.5 Data Analysis).

The theoretical framework that informed the development of a working typology for students spatial structuring of square grids are the works of Battista et al., 1998 and Clements et al., 2017. These two foundational studies have generated developmental typologies that map students' conceptions of square grids within a measurement perspective. However, this study focused on students' spatial reasoning, and required a fresh analysis of student understanding that placed spatial reasoning at the centre of the analysis focus.

A summary diagram illustrating the research design process is provided in Appendix A of this thesis.

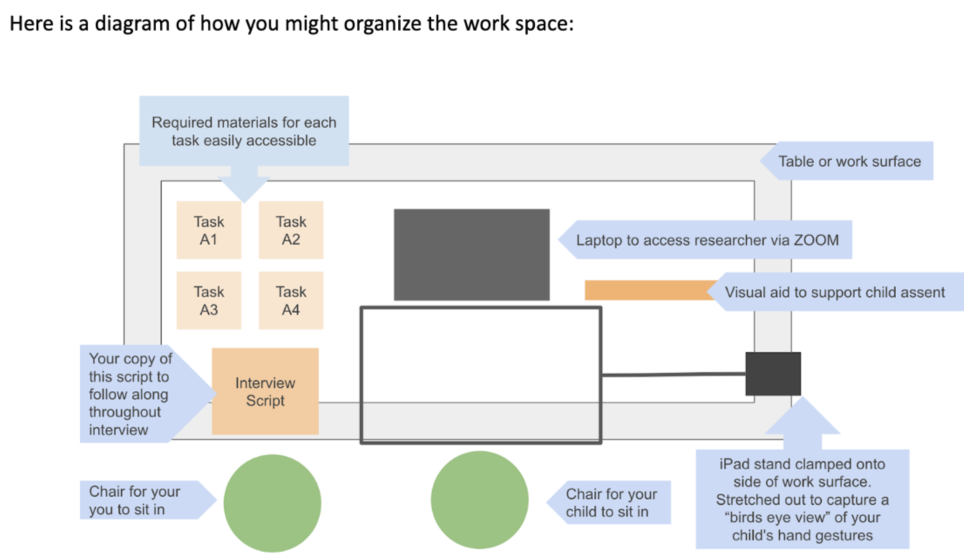
3.2 Context surrounding this study

The planning for this thesis began in the winter of 2019 and was re-imagined several times as the COVID-19 pandemic ground all in-school research to a halt. During the pandemic, it was not possible to work in-person with students due to the restrictions imposed by Public Health Units in Ontario, and subsequently school boards, which closed ethics application opportunities. Universities responded with increased protocols for in-person data collection. With these restrictions in place, it was clear that these interviews would have to be virtual and in the homes of families. Seeking and achieving ethics approval to do virtual interviews was not a simple task as there were significant restrictions around virtual interviews with students during this time. The task-based interviews required students to interact with physical materials (e.g., paper copies of grids, plastic tiles to cover grids). As a result, I had to devise a plan for distributing materials (ensuring students had task materials, as well as an iPad and iPad stand for filming students' interactions with materials) that would meet the increased safety requirements. I was able to create and rotate individual “kits” for distribution to families. This included an extensive protocol to wipe, clean and quarantine the kits in the trunk of my car for 72 hours before being dropped off on the subsequent families’ porches. The protocol can be found in Appendix B of this thesis.

Shifting to virtual interviews also required the cooperation of parents to take an active role in the interview process. I needed them to help distribute materials to their child at key points in the interview and to highlight certain features of grids for students with gestures as I described a task through ZOOM. Parents became co-interviewers with me sitting alongside their children virtually, throughout the interview. The coordination of parent and researcher throughout the interview process involved a number of directions for parents, including how they should set-up and organize the space for the interview. Below is a diagram provided to parents that showed a standard helpful arrangement for all of the task materials and electronic equipment in the kits (see Figure 11).

Figure 11.

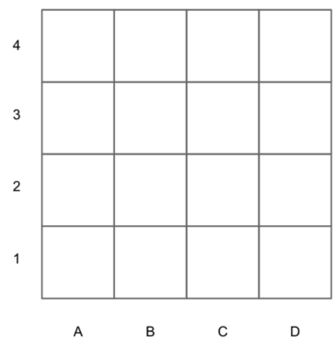
Instructions for parents on how to set up a workspace for the interview see Interview Script (Appendix C) for more information



To support parent and researcher alignment throughout the interview I provided a script that detailed what parents would say and do and what I would say and do for each task. Below is an excerpt, where the parent actions are featured in bold font.

Figure 12.

Excerpt from interview script showing parents actions



Researcher: “*Here is a grid. It has four columns”*
Parent uses a smooth sweeping gesture running their finger along each column from bottom to the top and says aloud A, B, C, D to highlight each column
Researcher: “*and it has four rows”*
Parent uses a smooth sweeping gesture running their finger along each row from left to right and says aloud 1, 2, 3, 4 to highlight each row
Researcher: “*Place a tile on C2”*
Parent hands child a plastic tile
(pause while student places tile)
Researcher: *“Thank you”*
Parent removes tile from grid

3.3 Participants

This study sought to undertake exploratory interviews with young children (ages 4-9) as participants. Due to the COVID-19 pandemic there were significant restrictions in place for accessing participants (e.g., schools were not open and school boards were not willing to consider ethics applications). Given these conditions a convenience sampling method was used to invite families to take part in the study. A convenience sampling method allows researchers to select participants who are “willing and available” to be involved in the study (Creswell, 2019, p.143). As an educator, mathematics consultant and parent in my community I had knowledge of several families with young children who were interested in their child's education. To obtain consent, I reached out to parents via email to see if they would consider participating in this study and if they would schedule a virtual meeting for information sharing. At the virtual meetings, I was able to explain the rationale for the study as well as detail the protocols that would be in place for the cleaning, distribution and quarantining of materials, as the COVID-19 restrictions were top of mind for families at this time. I was also able to explain the level of participation required by parents throughout the interviews. I am incredibly grateful that sixteen families agreed to participate in this study and that I was able to collect twenty-three virtual interviews from different children (see Table 2 below for age distribution of group). The greatest number of students in the participant pool were age 6. The least number of participants were at the extremes of the age band, namely ages 4 and 9.

Table 2.

Participant age range

Age of participants in years						
	Four	Five	Six	Seven	Eight	Nine
Number of participants (n=23)	2	4	6	4	4	3

3.4 Data collection

3.4.1 How data were collected

In order to adhere to the COVID-19 restrictions in place at the time of the data collection all interviews were conducted via ZOOM recording. While the interviews occurred via ZOOM, participants did have a chance to interact with physical materials and prompts. These included different sized grids and print materials. A kit of these materials and instructions were delivered to parents prior to the interview date. Parents supported the facilitation of the interviews by following a script (see Appendix C) and providing the materials to the child participants as needed throughout the interview. There were two video/audio recordings of each interview which necessitated that the families had internet access. One recording was captured through the built-in ZOOM recording feature of a laptop, and the other recording was captured through an iPad that was given to parents along with an iPad stand. Parents were instructed to angle the iPad over the child's tabletop area to get a "birds-eye view" of the student's hands as they engaged with the materials. The interviews were scheduled to take place at times that were convenient for families. The order of the tasks and wording was consistent throughout all participant interviews. The tasks were designed to be playful and engaging, with each of the five tasks taking approximately five minutes.

3.4.2 Tasks/Items used for this study

The following is a short description of each of the five grid-related tasks that students experienced during their interview. To see all the items in their entirety please refer to the Appendix D-H.

Task 1. Open questions about a square 4x4 grid (see Appendix D)

Materials Needed: Print out of 4x4 grid

Purpose: To gain insights into children's intuitive connections to grids

To begin, students were asked some general open-ended questions to get them talking about grids. Students were shown a 4x4 grid and were asked "what does this remind you of?" to gain a broad sense of their understanding of grid structures. A follow-up question asked children to be more specific and directed them to look at the grid more closely (perhaps by noticing lines, intersections and spaces that make up a square grid). Students were asked to generalize their understanding by considering some possible uses for grids.

Parent places 4x4 grid in front of child
Researcher: *"What does this remind you of?"*
(pause for student to respond)
Researcher: *"Look closely, what parts do you see?"*
(pause for student to respond)
Researcher: *"What could you use this for?"*
(pause for student to respond)
Parent removes grid
End of task

Task 2. Location on grids (see Appendix E)

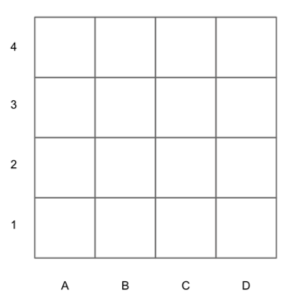
Materials Needed: 4x4 Location grid with labels, plastic tiles

Purpose: Explore how students locate and isolate intersecting spaces

The second task asked students to place a plastic tile on a specific location on a grid with the x and y coordinates labelled. The labels on this grid showed the letters A, B, C, D along the x-axis and the numbers 1, 2, 3, 4 along the y-axis. As the guide for the interviews, I directed the student's attention to these labels with some coordination between the parent and researcher. Particularly, the coordination was between the researcher's words and the parents' gestures that serve to highlight the specific area of the labelled grid for students. For example, when the researcher counted out each of the horizontal rows, the parent was asked to use a smooth sweeping motion along the row. This motion helped children see that the numbers labelling the row applied along the entire row. The same coordination was required for highlighting the columns. The direction of the parent's gesture was prescribed as well (e.g., bottom to top, left to right). The description of the labelled grid helped to orient children to the grid used in the task. All remaining prompts related to the task involved students placing a tile on a designated spot on a grid. Below is an example of what the task and coordination with parents looked like (note parents instructions are in bold):

Figure 13.

Coordination between parent and researcher actions



Researcher: "*Here is a grid. It has four columns*"
Parent uses a smooth sweeping gesture running their finger along each column from bottom to the top and says aloud A, B, C, D to highlight each column
Researcher: "*and it has four rows*"
Parent uses a smooth sweeping gesture running their finger along each row from left to right and says aloud 1, 2, 3, 4 to highlight each row
Researcher: "*Place a tile on C2*"
Parent hands child a plastic tile
(pause while student places tile)

Researcher: *"Thank you"*
Parent removes tile from grid

Task 3. Covered grid "blanket" task (see Appendix F)

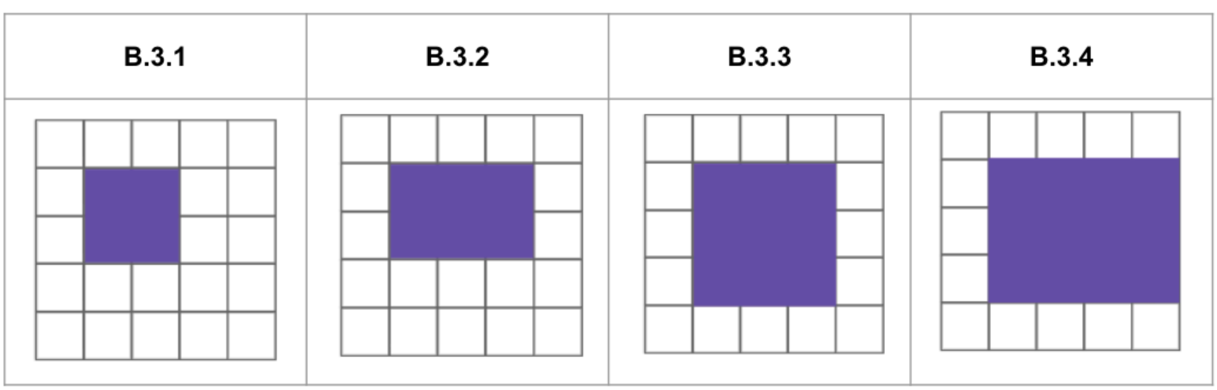
Materials Needed: Blanket array cards labelled B.3.1, B.3.2, B.3.3, B.3.4.

Purpose: Explore how students perceive unstructured space (supported with exterior grid lines)

This third task involved showing students a series of 5x5 grids. Each 5x5 grid had a rectangular section coloured purple covering up an area. Students were told that this smaller purple rectangle was a "blanket" and they were asked to determine how many of grids squares had been covered by the blanket. They were also prompted to explain their rationale for determining the number of squares. There were four questions within this task. Each blanket card was intentionally sequenced to get progressively more challenging for students. Below is an example of the sequence of blanket sizes students encountered:

Figure 14.

Covered area task



Parent places blanket grid card B.3.1 in front of child
Researcher says: *"Here is a 5x5 grid. There are 5 columns..."*

Parent uses a sweeping gesture from bottom to top to indicate each column
Researcher: "...*and there are 5 rows"*
Parent uses a sweeping gesture from left to right to indicate each row
Researcher: *"There is a blanket covering some of the squares on this grid. How many squares do you think are hiding under the blanket?"*
(pause for student response)
Researcher: "*How did you decide it was ___?"*
Parent removes grid B.3.1 and places B.3.2 in front of child

Task 4. Complete the grid (Adapted from Battista et. al., 1998) (see Appendix G)

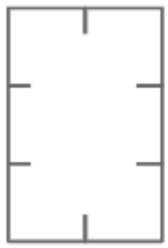
Materials Needed: Complete the grid cards B.4.1, B.4.2, pencil, plastic tiles

Purpose: Explore how students perceive unstructured space (tick mark support), explore how they construct grids through drawing

Students were presented with an image of a grid with missing interior grid lines. Some indicators are visible to help students see where grid lines should extend. The grids used in this task were scaled so that one cell covered a 1 inch by 1 inch area. The plastic tiles were 1 inch by 1 inch and covered one grid cell completely. Students were first asked to estimate how many plastic tiles they thought it would take to cover the entire grid and were then shown one plastic tile to use as a visual point of reference. Students were asked to "complete the grid" with a pencil, a prompt designed to see if they could visually extend the grid lines from edge to edge. If the student's drawing matched their prediction the researcher moved on to the next question. If there was a mismatch between a student's prediction and drawing, the child was prompted to check their thinking by covering the square grid area with the plastic tiles. Below is an example of a grid with missing grid lines:

Figure 15.

Example of 'complete the grid' task



Parent places grid B.4.1 in front of child
Researcher: "*Here is another grid, but look this one is missing some parts. How many square tiles would you need to cover this grid completely?"*
Parent holds up one square tile, if student tries to take the square tile the parent will say "try to picture it in your mind"
Researcher: Repeats prompt if needed "*How many square tiles would you need to cover this grid completely?"*
(pause for student response)
Researcher: "*Can you use a pencil to finish the grid?*
Parent hands student a pencil
Researcher: Repeats prompt if needed "*Can you finish the grid?*
(pause for student response)
Researcher: "*How many squares do you have?"*
If the response is correct, move to the next size grid.
If incorrect researcher asks: "*Let's check with the tiles"*

Task 5 Grid Comparison Task (see Appendix H)

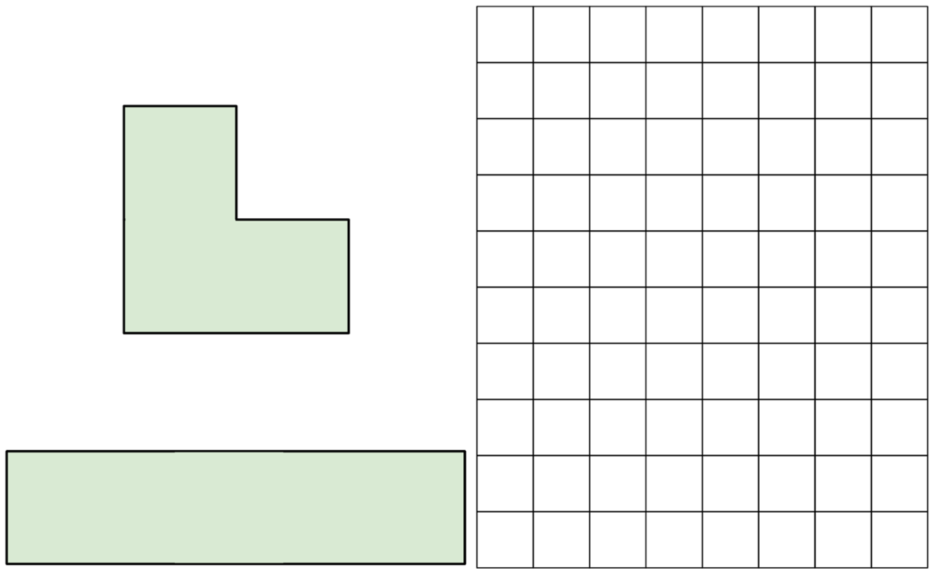
Materials Needed: Grid Comparison Task Cards B.4.1, B.4.2, B.4.3, B.4.4 and 2.5cm Grid Transparency

Purpose: Explore how students spontaneously use grids to support their reasoning

This task was designed to see if students would use grids to make a comparison between the areas of two different gardens. Students had available to them, within reach, an overlay of a grid transparency they could choose to use to help them make a comparison. The first question was designed as a practice question to help students understand the task. The remaining questions asked students to compare the areas of two gardens. Students could choose to use the transparent grid overlay to help with this comparison or use other invented strategies (e.g., estimation, mental rotation).

Figure 16.

Example of transparent grid comparing area task



Parent places task card B.4.1 in front of child
Researcher: *"This is a drawing of a garden. I want to know how big my garden is. How could you use this grid to tell how big the garden is?"*
Parent lays out grid transparency, it remains in child's reach for remainder of task
(pause for student to respond)
Researcher: repeats prompt *"How big is the garden?"*
(pause for student to respond)
Researcher: *"Here there are two different more gardens. Point to which garden is bigger?"*
(pause for student to respond)
Researcher: *"How do you know that one is bigger?*"

3.5 Data Analysis

There were twenty-three interviews for which I was able to collect video/audio recordings. Of the twenty-three interviews, two were discarded as parent involvement throughout the interview interfered with the students' responses.

As noted previously, this study used an ideal-type analysis approach which involved a specific sequence for data analysis as outlined by Stapley, O'Keeffe and

Midgley (2022) in their article ‘Developing Typologies in Qualitative Research: The Use of Ideal-type Analysis’. This process involves seven steps; Becoming familiarized with the data, Writing the case reconstructions, Constructing the ideal types, Identifying the optimal cases, Forming the idea type descriptions, Checking credibility, Making comparisons (see summary of actions in Table 3) (Stapley et al., 2022). I will now detail my examination of the data through the ideal-type analysis process.

Table 3.

Steps in Ideal-Types Analysis Process

Steps in Ideal-Type Analysis Process	**Actions Taken**
Step One: Becoming familiar with the data	- Reviewed researcher memos - Watched videos
Step Two: Case reconstructions	- Student-by-student analysis - Transcribed key moments - Coded data
Step Three: Constructing ideal-types	- Looked back student-by-student and assigned students to groups based on themes that emerged from coding - Summarized key features of groups as formed
Step Four: Identifying optimal cases	- Selected examples that highlighted key features of the group
Step Five: Forming ideal-type descriptions	- Looked at students within each group and summaries of key features - Formalized ideal-type descriptions
Step Six: Checking credibility	- Aligned groups created against existing research (see Table 2)
Step Seven: Making comparisons	- Analysis across groups looked for similarities & differences

Step One: Becoming familiar with the data set

I familiarized myself with the data set by; conducting the interviews, taking researcher memos, watching the videos, capturing screen grabs and marking time stamps of gestures captured.

Prior to analyzing the data I created a chart that showed the alignment of two major studies that served as the tested framework for my research; Battista et al., 1998 and Clements et al., 2017. I briefly summarized the different categories along their continuums and visually arranged these summarized categories to highlight areas that overlapped. This initial work prior to beginning my analysis allowed me to concisely describe the key features of each of the categories along their trajectories.

During and after each of the interviews, I took notes, in the form of researcher memos, on my first impressions and general thoughts. The researcher memos allowed me to capture initial impressions of the interview which I would later consider when determining which illustrative examples to select. These memos also allowed me to note significant and memorable events that occurred during the interviews. For example, there were times when students used gestures, gave explanations or created drawings that stood out as they were similar to the examples I had read about in Battista et al., 1998, Clements, 2017 and Outhred & Mitchelmore 2000. I noted my initial interpretation of the students' thinking (early or advanced conception of grids) and in which task the student demonstrated this thinking as a way to quickly capture what was unique about the interview from my first impressions.

Step Two: Case reconstructions

At this step in the analysis process, I re-watched the interview videos and wrote short summaries of what I believed was happening during the task-based interviews (when participants were interacting with grids). Only moments involving grids use and communications were transcribed in order to “focus solely on the sections of an interview transcript that are relevant to the study aim” (Stapley et al., 2022, p. 3).

During this step of analysis of the interview data I viewed each interview student by student. I created a spreadsheet to organize each student's responses across each of the five task items. As I viewed the video and listened to the audio, I transcribed the student's verbal explanations for each item in each task alongside their gestures (as the gestures and speech occurred simultaneously and were in support of each other). I also used the coding of “C” if students gave an appropriate (correct) response to an item. If students had an error in their thinking I would code that as “E”, with the student’s incorrect answer in brackets beside it. This allowed me to see at a glance, the individual items within a task that students were able to answer within reason as well as common errors or non-reasoned answers.

I developed a coding system to support tracking the different ways in which students were expressing their conceptions of grids (through their language, gesture and diagram). Coding the data helped me to name and track the differences I observed (Creswell, 2019). Through this first level student by student analysis, I created codes to identify the various kinds of gestures used (e.g., WH = whole hand, FP = finger pointing, SW = sweeping gesture, MS = pincher measure, BH = two-hand gesture etc.). I also noted the directionality of their gestures in relation to the student’s point of view (e.g., LtR = left to right, TtB = top to bottom etc.). I took screenshots of students' gestures and added these to the spreadsheet beside the column that listed their corresponding codes, annotations and timestamps.

Step Three: Constructing Ideal types

This is an important additional layer to the data analysis as it “leads to the formation of groups (or ideal types) of similar cases - participants with similar experiences or perspectives” (Stapley et al., 2022, p. 5). Through the practice of re-watching the videos and transcribing and reconstructing the cases (in Step Two) I had observed and coded key features in how students related to grids within each participant interview. Now, in this level of analysis I was able to consolidate my coded data and bring emerging themes together (Creswell, 2019). This involved assigning each participant to a group based on common features coded within the interview. To do this I went back through the spreadsheets and colour coded cells based on the recurring elements I had noticed (e.g., key words like “single cell” “tapping” “counting” were considered early conceptions and were colour coded as green etc.).

At first the extremes of the typology (early conceptions and advanced conceptions) were easiest to identify and to group together. Another group that stood out as unique was the group of students who noticed and used square units to support their structuring of grids. The remaining participants were more difficult to group initially. I had to re-watch carefully the interviews to determine which group they belonged to and thus created new groups that made sense for those participants. When no further types could be identified, and all participants had been assigned to a group, I was left with five distinct types of conceptions of grids that the children demonstrated (see Findings).

Step Four: Identifying optimal cases

Selecting which cases to use to illustrate each group along the typology required the specific characteristics of each type to be clearly defined in order to ensure that the cases selected were optimal representations of those types. As a guideline Stapley et al., (2022) explain that "optimal cases are those which most closely illustrate the pattern of similar cases that each group represents" (p. 5). This meant selecting moments from within a group that were representative of the defining characteristics of the group as a whole. Referencing my researcher memos during this stage helped to remind me of standout moments within interviews to re-watch and re-consider for this selection. I chose key moments from cases within a group that were considered as ideal-type examples and wrote up descriptive narratives and created visual renderings of those key moments.

Step Five: Forming the ideal-type descriptions

During this stage of the analysis, I needed to go back and look at the interviews within a group to ensure that the descriptions that formed those groups were representative of the group as a whole. This meant writing out a short summary description along with creating visual renderings to illustrate the key features of each type. When forming the ideal-type descriptions it was important that the description include "elements of other cases within the group as necessary to comprehensively describe the characteristics of that ideal type" (Stapley et al., 2022, p. 6). This meant that my ideal-type descriptions were not only describing the selected cases but instead were a description of this group based on evidence from across the interviews assigned to the group. The five types and related characteristics are included in the Findings section of this thesis.

Step Six: Checking credibility

When checking the credibility of the selection of cases "the aim of this step is to assess the clarity of the ideal types, rather than to ascertain the 'correctness' of the typology" (Stapley et al., 2022, p. 5). This involved checking to ensure the cases selected clearly aligned with the features assigned to the grouping. For this stage of the analysis I leaned on the area work of Battista et al., 1998 and Clements et al., 2017 to bolster the typology with a tested theoretical framework. I mapped my groups and their defining characteristics alongside their established continuums to see whether the groups generated through my data analysis were aligned. I could see similarities between the defining features of categories along their frameworks and the student interviews I had selected as representative of similar categories along my typology.

Step Seven: Making Comparisons:

The final stage of analysis required making comparisons both within and between types of groups in the data set to note similarities and differences (Stapley 2022). The analysis that looked *within* each group was previously generalized and summarized in *Step Five: Forming the ideal-type descriptions*. The findings of this step of analysis are featured in the Findings of this thesis.

Reflecting on the ideal-type analysis process

Stapley et al., explain that "ideal-type analysis provides a clear, rigorous, and systematic approach to constructing typologies using qualitative data" (2022, p. 7). The data analysis procedure for this thesis followed the process for ideal-type analysis as defined by Stapley et al., 2022. However, there were some differences in the process applied here that are worth noting. The design of this research involved asking participants a series of questions for each of the five tasks addressed within the virtual interviews. As expected, students did not respond similarly in all situations involving grids and demonstrated different characteristics at different moments across an interview. When students were grouped into types, decisions about which group to assign or create were made by looking at the interview as a whole and considering which characteristics were most consistently demonstrated throughout the entirety of the interview. When selecting ideal-type cases I have chosen key moments from within a group type that illustrate the group type characteristics, these could be from different cases within a group.

Also, when checking for credibility, the authors suggest having an outside researcher analyze the data to see if they would arrive at the same conclusions and assignment of groups when sorting the same data using the same criteria. This was not logistically feasible at the time of analysis and instead I have tested my categorizations by referencing back to the frameworks of Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2004 and Mulligan et al., 2020. While some adjustments were made to the ideal-type analysis procedure to make it applicable to the design of this research, the framework for analysis has proven to be a useful structure for analyzing qualitative interviews and generating a working typology. Stapley et al., acknowledges that “the steps to conducting an ideal-type analysis should not be considered inflexible but can be adapted by the researcher if necessary, according to the needs and nature of their study and data set" (2021, p. 79)

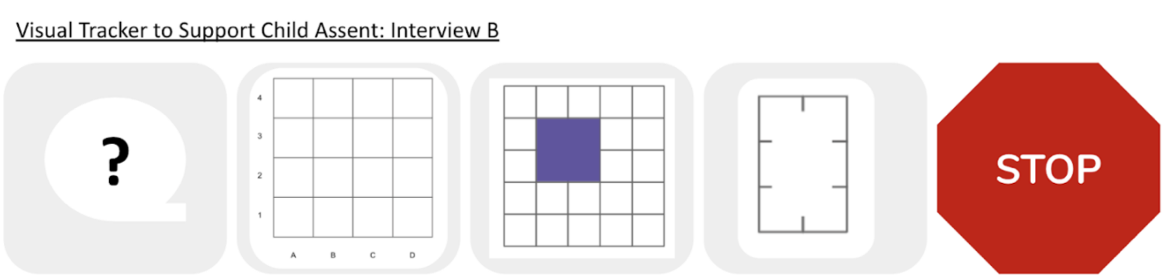
A summary diagram illustrating the ideal-type analysis process used in this thesis can be found in Appendix I.

3.6 Ethical Considerations

This study underwent a full ethical review and approvals and involved young children as participants. A primary concern for conducting research with this population involves parental consent but also the child's assent to participate in the study. I began each interview by describing the purpose and process of the study to the child participant in accessible child-friendly language and asked for their assent to participate. I created a visual progress bar to help the child see how many tasks were being asked of them and to keep track of their progress throughout the interview (see figure 12). The visual tracker showed a stop sign and I made sure to let the participants know that at any time they could stop the interview by pointing to the stop sign. I would refer to the visual tracker, featured in Figure 12, in between tasks allowing multiple opportunities to reaffirm the child's assent and provide an option to withdraw or pause the interview.

Figure 17.

Example of visual tracker used to monitor students assent to participate



The tasks designed for this study are age-appropriate to this population. Some were co-developed with classroom teachers using feedback and input from students derived from past Math for Young Children projects (Bruce, PI). Others are widely used assessment tools common in educational research specific to this population age range (e.g., “complete the grid” task adapted from Battista et al., 1998). Parents received a complete script detailing each task three days prior to the interview. Any tasks that parents deemed inappropriate for their child could be omitted.

Responses by student participants were audio and video recorded. This data is sensitive and was responsibly managed by the principal researcher. All electronic files were encrypted and password protected. The data was coded with a unique identifier for each participant and each task so that data files did not show students names.

The potential risks of participation in this study were minimal. There was a potential risk that the participating child could feel uncomfortable answering mathematics questions while being recorded and observed by the researcher and their parent. The parent might also feel uncomfortable with the way their child was responding to questions from the researcher. Parents and children were reassured that the aim of this study was to observe how young children think about grids without special preparation or teaching. The feedback from participating families suggested that in fact students enjoyed interacting with the mathematics tasks alongside their parents and parents also enjoyed learning more about how their child responded to mathematics-related questions.

Findings

4.1 General conceptions of grids

The opening task for each of the student interviews involved asking open-ended questions about square grids. Students were shown a 4x4 grid and asked “what does this remind you of?” and then as a follow up “what could we use this for?”. Students' general thoughts about grids presented in the image were fascinating. Many students related grids to things they were familiar with in their lives saying it reminded them of a waffle pattern or the pattern on a beaver’s tale. They also made connections to things at school saying it reminded them of a ten-frame or a hundred’s chart. Notably, many students said it reminded them of a Rubik's cube, connecting the two-dimensional grids they were shown to a tangible three-dimensional object they were familiar with in their lives. The 4x4 grid shown to students was scaled so that each cell measured one inch by one inch, this is very close in size to the individual cells on a Rubik’s cube (each cubelet is ¾ of an inch by ¾ of an inch). All children in the study were able to make a connection between the square grid presented before them, to their lived experiences.

When asked about the possible applications of grids students seemed less sure of what possible usages there could be for this structure. Some students said it could be used as a game board or a calendar. Another student said it could be used for counting, possibly thinking about the hundreds charts that are a common feature in most classrooms. In total twenty-one children were able to think of a use of the grid, suggesting that most children in the study were not connecting the grid presented to their potential use.

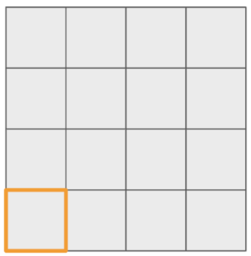
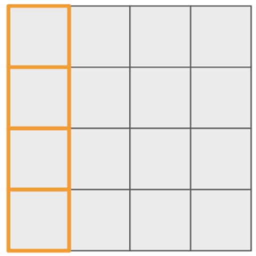
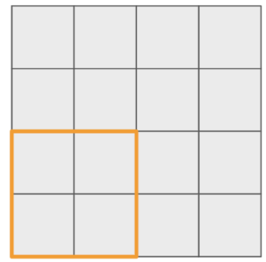
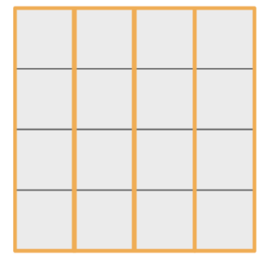
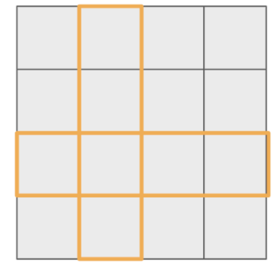
Given that this opening question was largely designed to orient the children, to help them feel comfortable, and to establish the use of technology and the provision of materials by parents, no further analyses were conducted on this question.

4.2 Constructing a Working Typology

The typology in Figure 18 was developed as a result of the ideal-type analysis of the twenty-one virtual interviews conducted in 2021. The typology aimed to describe different ways in which student’s interacted with and responded to grids. This typology used the work of Battista et al., 1998 and Clements et al., 2017 as a theoretical framework. This typology summarized students' conceptions of grids from a spatial perspective as opposed to considering them through a measurement context (see Figure 18).

Figure 18

Working typology of students spatial structuring of square grids

1. Single Cell Structuring	2. Partial Unit Building	3. Whole Figure and Parts-of-Figure Noticing	4. Composite Unit Structuring	5. Coordinated Structuring
				
Sees each cell as an individual unit with its own structure. The collection of cells are not bound by an overall system. The relationship between cells is non-continuous.	Sees an extended collection of the individual cells within rows and columns. Recognizes a combined relationship between cells in row or cells in a column.	Perceives the whole grid as well as its component parts such as rows and columns. Composes and decomposes shape isolating geometric figures within the grid. Conveys a sense of coordinated space.	Unitzes (groups) features of the grid to make sense of the larger structure. Isolates a unit (row or column) made up of individual units. Sees copies of the composite unit within the grid. Coordinates quantities (e.g., counts rows of four individual units as 1, 2, 3 4).	Perceives the coordination of multiple dimensions simultaneously. Recognizes a generalized relationship between rows and columns and their intersections.
- Taps each cell with pointer finger - May double tap when counting especially on cells whose structure is less discernible (e.g., corners, interior) - May count along a pathway supported by the visible structure of the perimeter of shape (e.g., spirals around towards interior where cells are less distinguishable) - May count all along row or column in a back and forth weaving pathway	- Fluid motion shows sequenced collection of individual units - May run finger along grid lines to help isolate individuals cells - May run finger along grid lines to partition space into cells - May run finger along centre of row/column to highlight collection - May trace an extended range of cells or grid line	- Frames geometric figure (usually with both hands or tracing of the perimeter) - Uses knowledge of geometric figures to make sense of spatial structure of the grid - Breaks down overall grid into recognizable geometric figures (mentally transforms, covers or copies figures) - Repeating geometric "parts" may or may not fill grid completely (e.g., square and half a square or square and 2 more in 2x6)	- Tap indicator when counting composite row/column units (e.g., whole hand marker, finger signifier, scaling pincher) - May use a scaling pincher gesture to measure the size of the unit being copied - Beat often rythmic to mark the unit counting	- Motions in both directions simultaneously to highlight row/column relationship - May use more robust gestures (e.g., both hands, use of fluid motion)

In this study, and based on the ideal-type analysis process, I was able to determine that there were five distinct types of perspectives and understandings of the grids. These were articulated through language, gesture and diagrams that the children used when asked to respond to prompts. The five types were as follows (reference Appendix J):

1. *Single Cell Structuring:* In this group children perceived square grids as being composed of a collection of individual cells, each with their own structure. The relationships between cells were therefore non-continuous . They did not see individual cells as bound to an overall structure.
2. *Partial Unit Building:* Children in this group perceived an extended collection of individual cells within rows and columns. They recognized a combined relationship between cells in a row or cells in a column.
3. *Whole Figure and Parts-of-Figure Noticing:* The children in this group conveyed a sense of coordinated space when they composed and decomposed geometric figures within the grid.
4. *Composite Unit Structuring*: This group unitzed (grouped) features of square grids together making sense of the larger structure. They isolated a unit (row or column) made up of individual units and saw copies of the composite unit within grids.
5. *Coordinated Structuring:* Children in this group perceived the coordination of multiple dimensions simultaneously. They recognized a generalized relationship between rows and columns and their intersections within square grids.

Illustrative examples for each of the five categories listed above were identified through the ideal-type data analysis process (see 3.5 Data Analysis).

When analyzing the data through the ideal-type analysis process, step seven required making comparison between types. Below are the findings from an analysis between groups (see Table 4).

Table 4.

Analysis between groups

Single Cell Structuring	Compared to "Coordinated Structuring" students often used individual finger tapping and in general less robust gestures were observed. A wide range of ages was observed in this type.
Partial Unit Building	Similar to "Single Cell Structuring" students were observed counting individual cells before they grouped the row or column as a unit. Students were generally observed to favour rows over columns similar to "Composite Unit Structuring".
Whole Figure and Parts-of-Figure Noticing	Often used more robust gestures (e.g., covered or tapped the figure with a whole hand unlike "Single Cell Structuring" which used an individual finger). A wide range of ages was observed in this type.
Composite Unit Structuring	Unlike students in the "Partial Unit Building" (who generally favoured rows) students in this type were observed selecting either rows or columns as a countable unit depending on the different grid sizes. A smooth motion running a finger along a row or column was observed in this group which differs from the individual cell tapping in the "Single Cell Structuring" type.
Coordinated Structuring	Students in the "Single Cell Structuring" were often observed double counting cells, whereas students in this type showed that a double count of a cell was part of row and column coordination. Older students were more often observed in this type of structuring.

4.3 Grid Typology: Illustrative examples

In this section, I will describe in detail select moments from the task-based ZOOM interviews that I believe illustrate the distinct types of student conceptions of grids as they relate to the working typology. In order to protect the anonymity of the student participants involved I have removed their names and replaced them with fictional names to make this section easier to read and to reference.

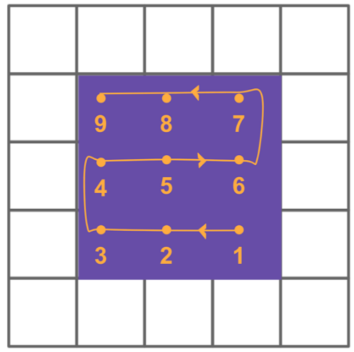
4.3.1 Single cell structuring

Students in this category perceived each cell within grids as individual units, each with its own individual structure, essentially an individual square by square perception. They did not engage with the cells on a grid as though they were part of an overarching structure and they did not appear to perceive any sort of continuous relationship between cells on a grid. An illustrative example of this kind of single cell structuring can be seen by closely observing the way Charlie (age 8) interacted with grids.

When presented with the covered area “blanket” task Charlie was given a 5x5 grid that showed a covered purple area of nine. He was asked how many squares he thought would cover that area and after some wait-time he responded “I’m taking an estimate, I think 10”. When describing where he “saw” the ten squares Charlie used his pointer finger to touch and name each individual cell as he counted. He started counting from the bottom cell on his right, tapping each cell one by one as he said the count, "one, two, three", moving from right to left across the bottom row. When he arrived at the end of the row, Charlie moved directly above (now moving left to right) and used the same tapping gesture with his pointer finger “four, five, six” as he moved across the middle row. He counted the final top row of the square area from right to left “seven, eight, nine” tapping each cell as he moved across the top row. The tapping and counting pattern that Charlie used can be described as weaving back and forth (see Figure 19).

Figure 19.

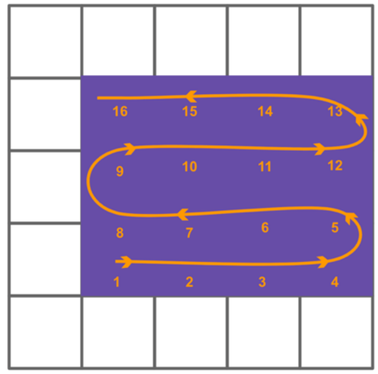
Charlie (age 8) used a single cell tapping and counting weaving pattern



On the next covered area task Charlie was shown a covered area of twelve on the same size 5x5 grid. This time the covered area reached the edge of the 5x5 grid, meaning there were no visible grid lines along one side of the area to support Charlie visually. Charlie gave an estimate, he said “maybe fifteen” when he looked at the covered area. It is interesting to note that he seemed less sure about giving his estimates on grids of nine and twelve as these areas had interior spaces that were more challenging to visually partition as they are not directly connected to the visible grid lines. To check his guess of fifteen Charlie repeated the same weaving technique he used earlier. This time he arrived at a count of sixteen (see Figure 20). His count weaved back and forth as before however this time he inaccurately counted an extra row of four. His rhythmic counting and tapping back and forth indicated he was counting a collection of cells and was not using or not able to extend the lines of the larger grid onto the covered area to guide him. Charlie’s use of a single cell tap and count in a weaving pattern is consistent with Battista’s one-dimensional pathway where students perceived grids as a collection of single units (Battista et. al., 1998).

Figure 20.

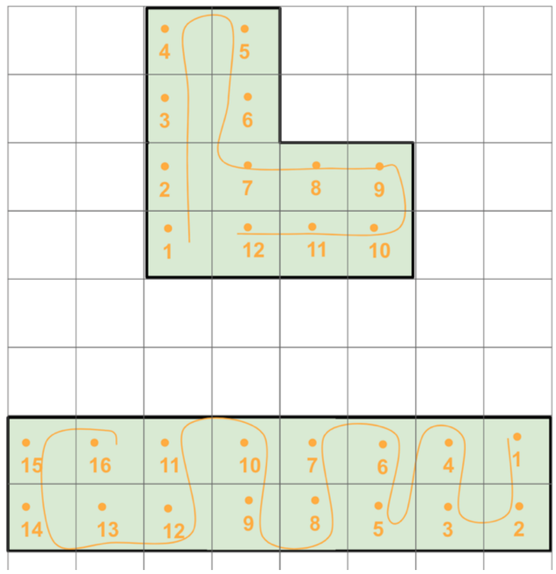
Charlie (age 8) used a weaving and pointing pattern to count the covered area indicating his conception of grids as an individual collection of cells, difficulty perceiving rows in unstructured space



As Charlie continued in the interview, he encountered a comparing area “garden” task (see Appendix H). In this task there were two “gardens” areas that were not presented on a grid, however a transparent grid was laid out and within reach if participants chose to use it to support their reasoning. Charlie was asked to decide which “garden” was larger. He looked at the areas and then reached for the transparent grid. When asked “how does that help you?” Charlie replied “you can count the squares”. He took time to carefully match the transparency to the areas so that the lines on this grid matched the edges of the areas. He began to tap and count each cell, this time his weaving pattern of counting was more serpentine and almost spiral in nature as he appeared to circle back towards his starting position (see Figure 21).

Figure 21.

Charlie (age 8) uses single cell tapping in a weaving and serpentine pattern

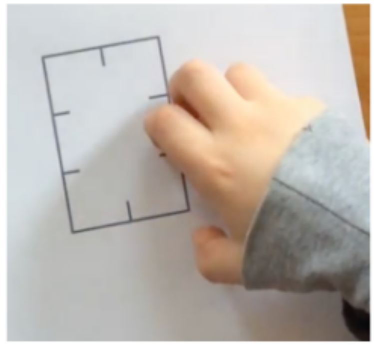


Charlie's use of an unsystematic method of tapping and counting individual cells indicated that he did not perceive a broader spatial structure of this grid but instead he saw a collection of items to be counted. His counting strategy changed when presented with different sized areas. He did not appear to perceive grids as having an inherent organizational spatial structure, instead he engaged with grids as though they were collections of isolated cells.

Another example of single cell structuring can be seen by looking closely at the way in which Robert (age 6) perceived grids. Robert was shown a 2x3 grid that was missing interior grid lines and was asked to predict how many square tiles would cover this particular grid. Robert curled his thumb and pointer finger and pinched them together to approximate the size of one tile (see Figure 22). His thumb and pointer finger were now able to act as a measure for the tile. Keeping his hand in this position, he tapped four times and said "four" as his count for how many tiles would fit. His taps did not appear to be directly correlated to the cells on his grid. He gave two taps in the middle and two taps in the lower area of his grid.

Figure 22.

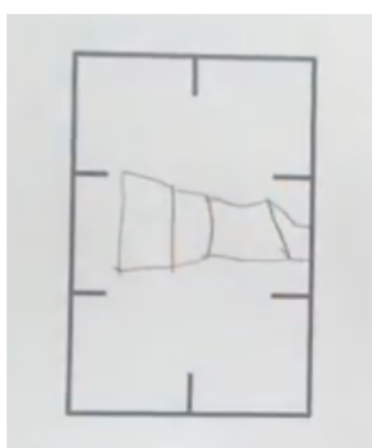
Robert (age 6) uses his thumb and pointer finger as a measure to approximate the area of a grid cell



Robert was then given a pencil and asked to “complete the grid”. He drew four square tiles from left to right across the middle row of his grid, not attending to our using grid line indicators along the border of his grid (see Figure 23). This task was the fourth grid task Robert experienced in the interview. In the three previous tasks he had been shown and asked to work with fulsome completed grids. He did appear to understand grids as a spatial object. Instead he is demonstrated that he perceived grids as an arrangement of individual squares.

Figure 23.

Robert (age 6) drew four individual squares when asked to “complete the grid



Robert was prompted to check his thinking using square tiles. He placed two tiles on his grid starting in the middle row on the far left side then another on the middle row on the right side (see Figure 24). He appeared to be trying to match his drawing to the square tiles by covering up his drawings with the tiles.

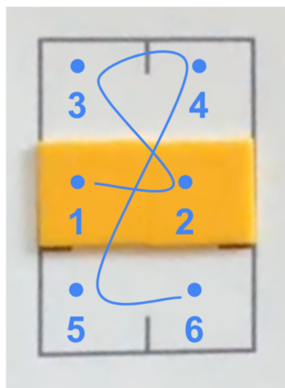
Figure 24.

Robert (age 6) covered his drawing with tiles before being able to isolate individual cells on a grid

As he covered the four squares he had drawn on his grid with two yellow tiles he said “two, four, wait, six, no not six, wait one, two, three, four, five, six, yes six!”. As Robert made his final count of six he used his pointer finger to touch and name each cell of his grid. It was only when the tiles were on his grid that Robert could isolate and touch and name each individual cell. He started his count by touching the far left cell on the middle row, then the cell beside that in the middle row, then the top left cell in the top row and then the cell beside that in the top row, then down to the bottom row counting left to right (see Figure 25). Robert started each count of a row on the left and then moved to the right each time. His choice to start in the middle could be because those cells were clearly marked by the tiles he placed on his grid. He did not seem to see any connection between how the cells were arranged on his grids. He touched and tapped each one individually as though they were six individual squares that happen to be arranged in an array. Robert did not see any relationship between the features of grids (rows and columns) and instead he interacted with grids as though each cell had its own unique structure.

Figure 25.

Robert (age 6) counting pattern example of student seeing grids as made up of disconnected isolated cells



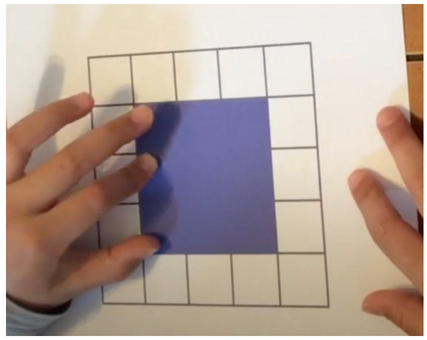
4.3.2 Partial unit building

The students who demonstrated reasoning related to this category saw grids as an extended collection of the individual cells within rows and columns. They were beginning to recognize a combined relationship between cells in a row or cells in a column. However, the partial units they recognized were disjointed and did not appear to have been generalized as part of the larger overarching grid structure. An example of this type of conception of grids can be illustrated by taking a closer look at how Faith (age 6) interacts with grids.

Faith was presented with a 5x5 grid with a purple covered area of nine and asked to estimate how many squares made up that covered area. She used a combination of finger tapping and her whole hand to help communicate her understanding of grids. Faith used her left hand and tapped her middle finger on the top left cell, pointer finger underneath on the far-left middle cell while her thumb covered the bottom left cell (see Figure 26). She used each finger to hold on to each of the cells in the column. This is different from the single cell isolators in the previous section who often used one finger to tap each cell individually in an unsystematic manner. Faith showed that she was beginning to group the isolated cells together. In fact, her whole hand was helping her to form and hold the whole column together as one unit. Faith picked up her whole hand and laid each finger down in the same corresponding spaces on the next two columns as she moved across from left to right. She gave a count of nine, when asked where she saw the nine she said "here, here and here" as she tapped her whole hand with the three fingers outstretched marking the cells within each column. Faith showed that each column is the same unit of three as her words "here, here and here" were in rhythm with her tapping of each column. When shown the next covered area of twelve Faith repeated the same gesture using the same three fingers and whole hand tapping across her grid.

Figure 26.

Faith (age 6) used each finger to match to individual cells and then tapped her hold hand three times across a grid from left to right as she said "here, here, here" indicating she is beginning to coordinate the count of a larger unit of three along with the number of individual cells that compose that unit (in this case three)

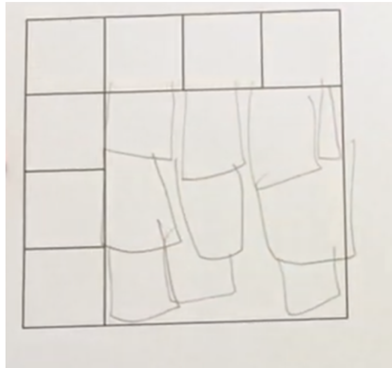


Faith was starting to see each column as a single unit, yet still marking each individual cell within that larger unit. Although she used her fingers to isolate each individual cell her conception of grids was different than that of a students who were single cell isolating as Faith was beginning to group the isolated cells together with her hand acting as a demonstration of both the one whole unit of three (her hand) and the individual grid cells that composed that unit (three fingers on one hand). By repeating this gesture across her grid along with her description of "here, here, here" she showed copies of a composite unit. She is in the early stages of seeing and building partial units on her grid.

Another example of this type of thinking (building partial units) on a grid can be shown in the drawings of Ada (age 5). Ada was given a 4x4 grid with a 3x3 section that was missing grid lines and asked to "complete the grid". Her drawing showed her building columns of a grid by copying individual cells downward. She drew each column all the way down and then moved onto the next column from left to right across her grid (see Figure 27).

Figure 27.

Ada (age 5) built columns by drawing cells downward



Ada showed she is starting to notice a column structure within the larger grid when she extended the marked columns downward drawing individual cells. She drew three sides to each cell (left, bottom, right) as she knew that the bottom line from the previous cell in a column would serve as the top of the cell below it. She made some errors (e.g., drew an extra cell on the far right and drew the bottom lines of the last cells in the column not connecting or using the line that was part of the larger grid). Ada's drawing showed that she is starting to see that there is some organizational structure to the ways the cells in a grid are connected spatially.

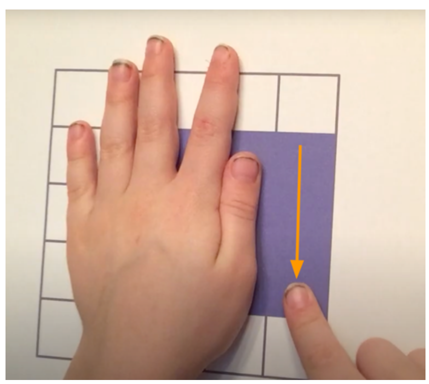
4.3.3 Whole figure and parts-of-figure noticing

When making sense of grids as a spatial structure some students isolated geometric figures within grids. They were able to both see grids as a whole while also breaking it into parts, such as rows and columns, as well as geometric figures within grids. In doing so they were demonstrating that they were beginning to coordinate the space (e.g., relationships between rows and columns). Roman (age 6) gave us an example of how students demonstrate their understanding of grids by isolating parts (in this case squares and columns).

Roman was shown a 5x5 grid with a covered area of twelve and would need to estimate how many squares comprised the covered area. He said twelve immediately before being prompted. Roman was asked how he just knew it was twelve so quickly. He responded “because I saw a square of nine here (traced the outline of the square accurately estimating where the extended grid lines would be then used his whole hand to cover and hold the square of nine) and if I cover up that there are three here (used his other hand to smoothly glide his finger down the column indicating the whole column is one unit of three) and I know nine plus three is twelve” (see Figure 28).

Figure 28.

Roman (age 6) used one hand to cover a square of nine and used his other hand to trace a column of three showing he decomposed the purple grid area into parts, specifically a square and a column



Roman is later shown the comparing area garden task and had a transparent grid within arms reach. He overlaid the transparent grid onto the area to support his comparison. He used a whole hand gesture to isolate squares of four across the shaded garden area. As he tapped each square, with his whole hand and all his fingers together, he counted by four “four, eight, twelve, sixteen” then moved to the other area and repeated the whole hand tapping of squares saying “four, eight, twelve”. He said “this one is bigger” and pointed to the larger area of sixteen. He explained “this one has sixteen, I counted by fours and this one has twelve, I counted by fours". Roman appeared comfortable both with counting in units of four and quickly seeing that unit of four as an isolated square within grids. He decomposed the larger grid into smaller square units of four and then used his whole hand to help him count in the square groups he had composed. Roman showed flexibility in being able to break apart the larger grids structure into smaller parts that suited his purpose.

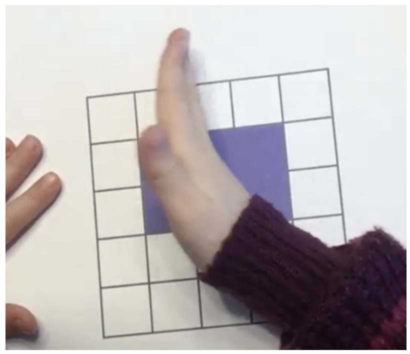
4.3.4 Composite unit structuring

Composite unit structuring occurs when a student shows they are beginning to group features of grids (rows or columns) and sees the larger structure of grids as made up of copies of those composite units. This requires students to coordinate quantities as they simultaneously track the number of units (rows or columns) and the count for the individual units that make up those rows or columns. An example of this way of thinking about grids can be illustrated through a closer look at how Poppy (age 7) interacted with grid tasks.

When engaged in the "Covered Area" task, Poppy (age 7), was shown a 5x5 grid that had a 2x3 purple shaded area and was asked "how many squares do you think would fit under the blanket?". Poppy responded quickly with "six" and when asked to elaborate on how she knew it was six she replied "because two, four, six". Poppy used a whole hand gesture that she tapped along each column moving from left to right and her count of "two, four, six" was coordinated with the tapping of her whole hand on each column. Using her whole hand to tap each column, as opposed to using two fingers to show each individual unit within the column, indicated Poppy had generalized the unit of two she was now counting in. Poppy saw the whole column as a unit of two and used her whole hand gesture to track how many groups of that unit of two she had for a total count of six (see Figure 29).

Figure 29.

Poppy (age 7) used a whole hand tapping gesture as a composite unit indicator



When Poppy was presented with different sized covered areas she continued to demonstrate that she could isolate composite units (each time she preferred isolating columns). She matched the tapping gestures she used to track her composite units with the rhythmic beat of her counting. When shown the 3x3 covered area she quickly said “nine” and tapped with her finger at the top of each column “three, six, nine” and used the same strategy again saying “three, six, nine, twelve” when shown the 3x4 covered area.

As the interview continued, Poppy encountered a new task. This time she was shown a series of grids with missing parts and was asked to imagine how many tiles it would take to cover the space. She was shown a 3x3 grid where the interior connecting grid lines have been removed. Poppy made an accurate prediction of nine. She said “three, six, nine” as she coordinated her count with her gesture indicating that she saw units of three, this time she tapped each row as a group of three. Poppy was able to isolate parts of her grid (either rows or columns) into units she was comfortable counting in (in this case rows of three).

When Poppy was asked to “complete the grid” she quickly and easily connected the missing grid lines. Interestingly, she rotated the page each time when connecting grid lines indicating flexibility with how rows and columns are constructed.

4.3.5 Coordinated structuring

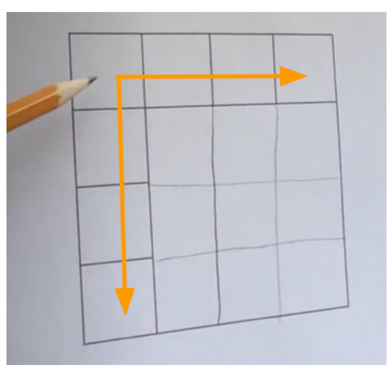
Students who demonstrated this type of spatial conception of grids perceived the coordination of multiple dimensions simultaneously. They showed evidence of recognizing a generalized relationship between rows and columns and their intersections. An illustrative example of this type of coordinated structuring can be seen through examining how Winnie (age 9) perceived grids.

Winnie was shown a 5x5 grid with a covered area of twelve and was asked to estimate how many squares would comprise the covered area. She said twelve and explained “the length is the same as four squares” as she tapped the page below the bottom row of the area. She continued “and this is three squares” while tapping to the far-left side of the covered area to indicate the column, “and four times three is twelve”. Winnie has given a count of the bottom row and far left column and coordinated that count by multiplying them together to give a total amount for the covered area. She doesn’t need to tap and touch each cell or row and column across grids as she recognized that the count for one row and one column could be multiplied together to give the count for the total area. She made sense of the overlapping spaces on grids knowing that a count of the far-left bottom cell as part of the last row and again as part of the far left column was needed to make her calculation and did not represent a double count.

Later when shown a 4x4 grid with a missing interior 3x3 area Winnie quickly and easily extended grid lines across the empty space. She used her pencil as an extension of her finger and gestured with the pencil as she said “each of these rows is four” . The pencil began at the far top left cell and hovered above the cells, then she swept it along the top row. She continued “and four times four is sixteen” as she said that the pencil returned to the far top left starting cell and swept down the column (see Figure 30). Winnie’s coordinated gesture (moving the pencil across one row then down one column) and her matching explanation “four times four is sixteen” indicated that she had coordinated her understanding of the larger grid structure. Not only could she isolate rows and columns, she could see rows and columns as related in a specific way with overlapping space.

Figure 30.

Winnie (age 9) used a pencil to gesture the coordination of rows and columns



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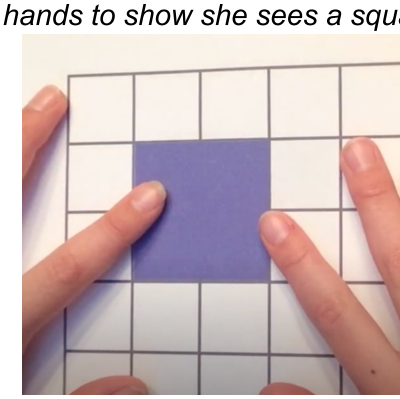
4.4 Seeing squares

One interesting finding that became apparent from analyzing the student interviews was the number of students who seemed to easily recognize and unitize squares on grids (e.g., squares of four and nine). When students can immediately visualize a familiar image, form or shape it is sometimes referred to as “gestalt”, the child's “effort to establish order by imposing familiar structure on unfamiliar things” (Moss et al., 2016, p.182). When looking at a large grid some students were able to make sense of that structure by recognizing a familiar shape, in this case a square. Students also seemed to have connected their knowledge of that shape with the quantity that comprised the square (e.g., a square of four squares, a square of nine squares).

When Poppy (age 7) described how she knew immediately that a 2x2 covered area on a larger 5x5 grid had an area of four she replied “because one is a square and if you add three more it's a bigger square”. On the same task Amy (age 8) said she knew it was four “because of the shape”. When giving this response Amy used both hands to tap the square indicating she saw the whole square of four as one structure (see Figure 31). When Amy moved to the next item in this task she was shown a 2x3 covered area. She quickly gave a response of six and ran her hand along the additional two squares and said “I knew it was two more”.

Figure 31.

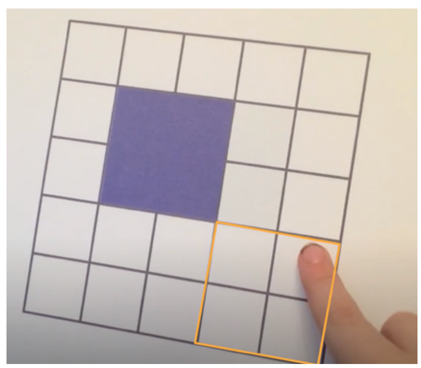
Amy (age 8) gestured with both hands to show she sees a square of 4



When Roman (age 6) was shown the same 2x2 covered area his explanation of how he knew it was four also involved him seeing squares on his grid. In this case however, Roman isolated a similar sized square on his grid outside of the covered space and was able to pick up, move and match the two areas in his mind's eye. He said “it’s the same size as the other spots where there is four” as he quickly traced around and around the squares on the larger uncovered grid space showing that the size of the two areas were a match (see Figure 32).

Figure 32.

Roman (age 6) visually saw and matched the area of squares on a grid



When Roman was shown the 3x3 item in the covered area task he said “that’s nine!” quickly and confidently before even being prompted to give a response. He was asked how he knew so quickly that it was nine and he replied “because I have a 3x3 Rubik’s cube and it’s the same size!”. The square arrangement of numbers on a grid (which in these interviews are 2x2 and 3x3) were immediately obvious units for Roman. He saw the squares as a familiar group and more importantly he saw the shape and number as inseparable.

When looking at a 3x3 grid without any grid lines Evan (age 8) said “it’s nine because it’s an even square”. He continued to see nine as a square when given the next 3x4 rectangular covered area of twelve saying “well it’s nine, with three more”. There were many examples like these throughout the interviews where students would name the shape and quantity together. Students then demonstrated that being able to quickly perceive the shape of a square area on a grid was useful when problem solving.

Discussion

5.1 Summary

This thesis endeavored to explore how young children (ages 4-9) think about grids as spatial objects in mathematics. Two research questions were considered:

1. How do young children perceive the spatial structure of two-dimensional square grids?
2. What drawings, gestures and language correspond with students' conceptions of square grids?

Question #1

The first research question focused on *how* students spatially structured square grids. Interviewing students as they engaged in different grid-related tasks resulted in many different *types* of student perceptions. The results of the data analyzed in this study align with existing research that demonstrates a wide range of student conceptions of grids, and revealing how complex the spatial features of grids are for young children to consider (Barrett et al., 2017, Battista 1999, Battista et al., 1998, Battista & Clements 1996, Clements et al., 2017, Outhred & Mitchelmore, 1992, Outhred & Mitchelmore 2000, Sarama et al., 2003). The range of student conceptions, gathered in this study, were mapped along a typology of spatial conceptions of grids that likewise aligns with the findings of spatial structuring of grids as observed within measurement contexts (Battista et al., 1998, Clements et al., 2017).

There were five types of conceptions of grids that were observed within the limited data collected in this study. These five types were;

1. *Single Cell Structuring:* In this category, children perceived square grids as being composed of a collection of individual cells, each with their own structure. The relationships between cells were therefore non-continuous. They did not see individual cells as bound to an overall structure.
2. *Partial Unit Building:* In this category, children perceived an extended collection of individual cells within rows and columns. They recognized a combined relationship between cells in a row or cells in a column.
3. *Whole Figure and Parts-of-Figure Noticing:* In this category, children conveyed a sense of coordinated space when they composed and decomposed geometric figures within the grid.
4. *Composite Unit Structuring*: In this category, children unitized (grouped) features of square grids together making sense of the larger structure. They isolated a unit (row or column) made up of individual units and saw copies of the composite unit within grids.
5. *Coordinated Structuring:* In this category, children perceived the coordination of multiple dimensions simultaneously. They recognized a generalized relationship between rows and columns and their intersections within square grids.

While overall, younger children were more likely to align with earlier conceptions of the grid, there were many children who presented across the typology at different ages. Age did not appear to be a key determinant for how children perceived grids, given the limited data set. This is consistent with the Battista et al.,1998 initial study of 123 children from Grade 3 and Grade 5 where age did not determine where children organized along the continuum. Their investigation described Grade 3 students demonstrating sophisticated coordinating of arrays and likewise Grade 5 students with early, less coordinated constructions. The authors noted that how students perceived the spatial structure of grids was highly individualized, and that is what this study found as well: children uniquely responded to the tasks and varied within these responses in terms of how they were thinking (across types) (Battista et al., 1998).

<u>Question #2</u>

The second research question focused on determining what kinds of drawings, language and gestures were associated with different conceptions of grids. The lists below are summaries of what was observed within the limited data collected in this study. It is important to note that the analysis of drawings, gestures and language was not exhaustive and instead served as providing multiple information points into how students spatially structured grids. The following summary identifies the kinds of drawings, gestures and language that were observed as it relates to the overall typology.

1. *Single Cell Structuring:*
    1. Often taps each cell, usually with a pointer finger
    2. May double tap when counting especially on cells whose structure is less discernible (e.g., corners, interior)
    3. May count along a pathway supported by the visible structure of the perimeter of shape (e.g., spirals around towards the interior where cells are less distinguishable)
    4. May count all along a row or column in a back and forth weaving pathway
    5. May count aloud by ones “1,2,3…”
    6. May draw individual cells, especially all four sides of each cell, cells are often disconnected, may draw in a spiral or disorganized manner

2. *Partial Unit Building:*
    1. Fluid motion shows sequenced collection of individual units
    2. May run fingers along grid lines to help isolate individuals cells
    3. May run finger along grid lines to partition space into cells
    4. May run finger along centre of row/column to highlight collection
    5. May trace or cover up with fingers an extended range of cells or grid line
    6. May count aloud along a unit by ones “1,2,3…” may also repeat aloud the count for the unit “3,3,3”

7. May draw some connected cells in a row or coloumn

3. Whole Figure or Parts-of Figure Noticing

a. Frames geometric figure (usually with both hands or whole hand, sometimes traces the perimeter)

b. Taps with whole hand or points to centre of figures to track copies of figures that cover grid

c. Covers with whole hand or frames geometric area (repeating geometric "parts" may or may not fill grid completely (e.g., square and half a square or square and 2 more in 2x6)

d. May count aloud in the counting unit "4, 8, 12", or give a count of units "1,2,3"

a. Draws extending lines in either direction, may rotate grid

4. Composite Unit Structuring

a. Tap indicator when counting composite row/column units (e.g., whole hand marker, finger signifier, scaling pincher)

b. May use a scaling pincher gesture to measure the size of the unit being copied

c. Beat often rythmic to mark the unit counting

d. May count aloud along either rows and or columns

e. Usually draws copies of rows or columns

5. Coordinated Structuring

b. Motions in both directions simultaneously to highlight row/column relationship

c. May use more robust gestures (e.g., both hands, use of fluid motion)

d. May count aloud along either rows and or columns

e. Draws extending lines in either direction, may rotate grid

Taken together, the above lists clearly illustrate how language, gestures and drawings were used by the children in the study to convey meaning – to convey their spatial reasoning about grids and their understanding of grids in a multi-modal fashion.

5.2 Significance of the study

This is a small but important study. Previous mathematics education research on grids has largely focused on area measurement concepts. This study draws attention to grids as mathematical objects that can help children make sense of space and reason spatially, with potential reach across multiple mathematics contexts. Researchers in the SRSG assert that "spatial reasoning should be developed *for the sake of better spatial reasoning* – that is, not just in the service of better number sense or in anticipation of its utility for teaching algebra" (Davis et al., 2015, p. 146). There is a need for research into student conceptions of grids as spatial objects in and of themselves. It is hoped that this thesis will bring attention to the ubiquity of grids throughout mathematics classrooms and mathematics education research and similarly draw attention to the under exploration of grids as a spatial objects worthy of study.

This study also analysed gesture, language and drawings together as a multimodal perspective on student thinking about grids as spatial objects, challenging the notion of separating this analysis. Researchers are inviting "us to avoid the implied dichotomy, preferring instead to view diagrams (and speech, gestures and other actions) as the active space for thinking itself" (Sinclair & Bruce 2015, p.322). This thesis attempted to view children's gestures, language and drawings as a part of their active thinking about grids as spatial objects.

5.3 Limitations

There are several notable limitations to the data collected in this study. First and foremost this data was collected during the COVID-19 pandemic. At the time of this data collection we were experiencing stay-at-home orders in our province and were in between the second and third wave of the pandemic. It was a time of uncertainty and anxiety for families and it is understandable that the context of these times may have impacted students' level of attention and engagement.

Secondly, as a result of the fluctuating COVID-19 restrictions these interviews had to be conducted virtually. As a result of the distance created by the virtual constraints, there were times when it was hard for me to hear and understand what students were saying. Consequently, by not being in the same room as students it was difficult to prompt or give feedback that was timely and could have potentially increased the communication from student participants.

A third limitation is the size of the participant group. Some would consider this sample size to be too small to make large claims. I concur, and in fact the study was aimed at developing a draft typology of how children “saw” and “thought about” grids. The sample size was sufficient to achieve saturation for the typology. That is, the interviews proved useful in developing a comprehensive typology and no new types or categories could be created, as the types began to repeat consistently with the participant pool in the age bracket of the study. In moving to older aged students it is possible that other types might emerge.

Lastly, as a result of the pandemic restrictions and virtual nature of the interviews I had to ask parents to support the students throughout the interview process. Their presence undoubtedly impacted how students felt about the tasks and responded to the tasks. Doing a math task while being filmed and sitting beside your parent places additional stress on the student participants. There were a few cases where parents would go off script, give materials that weren’t applicable to the task or provide prompts for their child which caused some interviews to be unusable. I am incredibly grateful to the generosity of the parents who made it possible to capture important information about how students engage with and think about square grids given the circumstances.

5.4 Implications

The results of my analysis, and review of the literature, suggest that the ways in which students spatially structure grids is wide ranging and complex. There is a need for further research looking at grids as spatial objects. There is also potential for classroom embedded research that aims to develop practical and engaging experiences for students related to grids. This section will suggest a series of implications for further research as well as suggestions for classroom practice.

5.4.1 Implications for further research

There are two areas of potential research that could further the work of viewing grids as spatial objects in mathematics education research. The two areas for potential research described below are:

- Spatial objects-to-think-with that span mathematical concepts
- Dynamic grids as spatial mathematical objects

Spatial objects-to-think-with that span mathematical concepts

Much of what students experience in elementary education involves discrete counting of quantities, with very little attention to spatial structures. Grids, on the other hand, instantiate concepts of continuous space and co-relational thinking (considering the coordinated intersections of rows and columns). As noted throughout the literature review, grids underly many of the mathematical concepts students explore in class with little explicit attention to their spatial features. Two notable examples of classroom applications that often involve unexamined grid structures are fractions and multiplicative thinking.

Fractions are notoriously difficult concepts for students to learn and for teachers to teach (Bruce et al., 2019). Students represent, compare and operate on fractions with support of (unexamined) grid structures. Thinking about fractions often involves thinking about coordinating relationships within and between quantities (e.g., when reasoning with fractions students consider the relationship between the numerator and the denominator as opposed to concrete whole number thinking). Similarly, multiplicative reasoning asks students to shift away from counting discrete whole numbers and to simultaneously negotiate the size of groups with how many are in each group. I have observed an abundance of (unexamined) grid structures within multiplicative reasoning applications in Ontario classrooms.

Paying attention to grid structures has the potential to support students reasoning across mathematics contexts, such as fractions or multiplicative reasoning. Mulligan, Oslington and English (2020), have reported that by drawing students' attention towards a grid's spatial structure and encouraging students to make generalizations they have observed growth in other areas of mathematics. The Australian researchers reported that “pattern and structure were inextricably linked with multiplicative reasoning, i.e., establishing the notion of composites, the unit of repeat, and partitioning” (Mulligan et al., 2020, p. 664). The researchers described students' increased ability to spatially structure arrays and ten frames (through PASMAP) as supporting the development of students' multiplicative and algebraic reasoning (Mulligan et al., 2020). While there is a lot we do not know about how students' spatial structuring of grids impacts their reasoning across other mathematical concepts, there is promise and possibility for grids to play an important role in supporting mathematical thinking beyond the current and dominant research focus of area measurement. There is a need for further research into how students think-with grids in novel contexts in mathematics.

Dynamic grids as spatial mathematical objects

This study was limited to static square grids. There is great potential for further research into how experiences with dynamic and mutable grids impact the ways children make sense of grids. As noted in the data analyzed in this thesis, and the spatial structuring of rectangular grids (Battista et al., 1998, Clements et al., 2017, Mulligan & Mitchelmore 2004, Outhred & Mitchelmore 2000), many early conceptions of grids involve “seeing” a collection of individual squares not bound to any larger overarching structure. There was a presumption underlying students' classroom experiences with grids, that grids were inherently static. This is consistent with instruction in Canada and the U.S which often presents mathematical concepts as static, neglecting the potential for leveraging spatial thinking through the dynamic movement of objects and bodies (Bruce & Hawes, 2015). Researchers have begun to explore the positive impacts that dynamic movement has on students' understanding of spatial concepts (Bruce & Hawes, 2015). Let's imagine if children could pull and move grid lines and "see" the impact of that movement on the larger grid structure, perhaps that could support them in generalizing the spatial features of grids and in applying these dynamic grids to dynamic contexts.

5.4.2 Implications for classroom practice

As a result of my investigation of the literature as well as my analysis of the data in this study, combined with my years of experience supporting student learning, I have purposed four potential implications for classroom practice. In this section I have described the four areas that I feel have potential for classroom applications with students in the aim of supporting their spatial reasoning with grids. These four areas are as follows:

- Drawing as part of active spatial thinking
- Squares as inherent structures of number and space
- Considering a range of grid types
- Grids as objects-to-think-with

Drawing as part of active spatial thinking

In this study, there were often differences in how children communicated their understanding of grids through language and gesture and what they ultimately drew when prompted to complete the grid. In early investigations into students' perceptions of grids researchers noted that to cover an array with tiles or to construct an array using tiles "seemed a trivial task compared to its inverse" (Outhred & Mitchelmore, 1992, p.

535). The inverse (constructing the grid one's self) was shown to be significantly more difficult because to draw the array required students to analyze and perceive its structure more generally (Outhred & Mitchelmore 1992).

Drawing is connected to visual spatial skills and "recent evidence suggests that drawing activities might also be an effective way of improving young children's spatial reasoning" (Hawes et al., 2015, p. 39). Spatial drawing, supported through quick drawing activities, has been shown to improve students' spatial skills (Moss et al., 2016, Tzuriel & Egozi, 2010). Similarly, drawing different arrays and reflecting on what students imagined and what they drew has been a successful part of the Pattern and Awareness of Mathematical and Structures Program (Mulligan & Mitchelmore 2009, Mulligan et al., 2020). There may be ways to playfully engage children in drawing grids that help them attend to the spatial features of grids in their constructions. One possibility is to build upon the "Can You Draw This?" spatial task described in book *Taking Shape: Activities to Develop Geometric and Spatial Thinking Grade K-2* (Moss et al., 2016). The "Can You Draw This?" activity asked students to quickly sketch a spatial object that was briefly presented to them (flashed on a projected screen or a piece of paper). This required students to hold the flashed image in their minds-eye before they translated that image onto their sketch. One example of an adaptation to this task using grids could invite students to quickly draw different grids shown in a strategic sequence, arranged to help students notice spatial features of grids (e.g., 2x2, 3x2, 4x2, 5x2 or 2x2, 3x3, 4x4 5x5 etc). Shared discussions of strategies used, after students sketch their grids, could support collectively building capacity as a group, helping make the spatial features of grids explicit to all students. It would be laboursome for students to draw individual cells. A playful drawing activity could necessitate generalizing the structure of grids to make drawing easier.

Squares: Inherent structure of number and shape

In the Findings section on seeing squares (see Section 4.4) the gestalt of squares on grids was observed in the data collected in this study. Squares seemed to jump out to some students and that supported their reasoning, as the square shape and the quantity were perceived as inseparable. Students informally bump into square numbers in elementary school. In our current 2020 Ontario Mathematics Curriculum, buried inside the Grade 4 Patterns & Relationships strand (C1.3) a sample task (#5) asks students to find the sequence of square numbers (although it does not mention building the corresponding arrays). It isn't until Grade 7 that students are formally introduced to square numbers in Ontario (B1.2 Rational Numbers). Despite the lack of attention to square numbers in our Ontario curriculum young children (age 4) were observed in this data set to intuitively recognize squares.

Through the PASMAP in Australia young children (4-5 years old) were intentionally prompted to make and draw from memory square numbers and the patterns they notice (Mulligan et al., 2020). Some of the questions researchers asked young children were: "What comes next in the pattern so it is getting bigger each time? Can you continue the pattern and work out the tenth largest square? What number patterns can you see?" (Mulligan et al., 2020, p. 672). Researchers reported that through these prompts and attention to building square numbers students developed their sense of "number patterns, multiplication and commutativity emerged as well as area measurement" (Mulligan et al., 2020, p. 672). Squares were observed to be intuitive and immediately recognizable to some students in this study. Are we missing opportunities in Canada to help students connect the shape and quantity of numbers through playful experiences building and playing with squares? We have an opportunity to make the building, finding and drawing of squares a playful discovery in classrooms before students ever need to think formally about square numbers. Students might be supported in their structuring of grids through noticing the patterns that emerge when constructing squares.

Considering a range grid types

We have discussed the predominantly static nature of grids in elementary classrooms. The grids students see in classrooms are not only static with fixed perspectives they are also most likely to be square grids. Broadening the range of types of grids students are exposed to could support them in generalizing the spatial features of grids and applying them to a wider range of contexts. If we want students to be able to imagine grids in their mind's eye and adapt grids to suit the problems they are solving (to construct grids as an object-to-think-with) one way we might support this is to provide experiences with a range of grid types. There are many possibilities of grid types. A few examples that might be easy created in the classroom are triangular grids or hexagonal grids. Early conceptions of grids involve perceiving grids as a collection of individual squares (Battista et al., 1998, Battista 1999, Clements et al., 2017, Outhred & Mitchelmore 2000). Therefore, by building, drawing, playing with triangular or hexagonal grids (or any tessellating shapes) could help students to begin to generalize the spatial structures of grids. Perhaps building other shaped grids could help students uncover an important feature of the square grids they are more familiar with - the fact that square grids only result in square shaped cells because of the unique perpendicular way that grid lines intersect. They could create many different types of grids through many other kinds of tessellations.

Viewing a range of grids from a range of perspectives would also support students in broadening their conceptions of the spatial structures of grids. Susan Gerofsky noted that students and teachers were using their whole bodies to think about and communicate their understandings of grids (Gerofsky 2011). Students could explore moving their bodies along the grid (e.g., walking along grid lines built with masking tape along the floor). Students could even imagine themselves as being the grid (e.g., the vertical midline of our bodies as the y-axis and the horizontal midline of our bodies as the x-axis) and feel movement and distance in their actions. These are just a few possible ways we might invite students to experience different perspectives of grids.

Grids as objects-to-think-with

Perhaps even more relevant would be to purposely design playful tasks that would benefit from spatial grid thinking, where children can begin to apply grids as they wish (imaginatively or physically) without prompting. There are endless playful tasks that would benefit from spatial grid thinking, such as tasks involving the list of actions generated by the SRSG; “locating, orienting, decomposing/recomposing, balancing, patterning diagramming, navigating, comparing, scaling, transforming and seeing symmetry” (Bruce et al., 2017, p. 146). Imagine if students were engaged in playful tasks that had an inherent need for spatial grid thinking (or would be strongly enhanced through the use of grids). Now imagine that those same students had experienced a range of opportunities to playfully construct a relationship with grid structures (through experiences with dynamic grids, a range of grid types and explicit conversations about grids as spatial objects). Would those students be more likely to imagine grids, construct physical grids, use dynamic grid software or transparent grids, embody grids with their gestures and movements? In this imagined space, grids have the potential to truly become an object-to-think-with for students, a lifelong tool for thinking. The joint work of mathematics education researchers and educators holds much promise moving forward.

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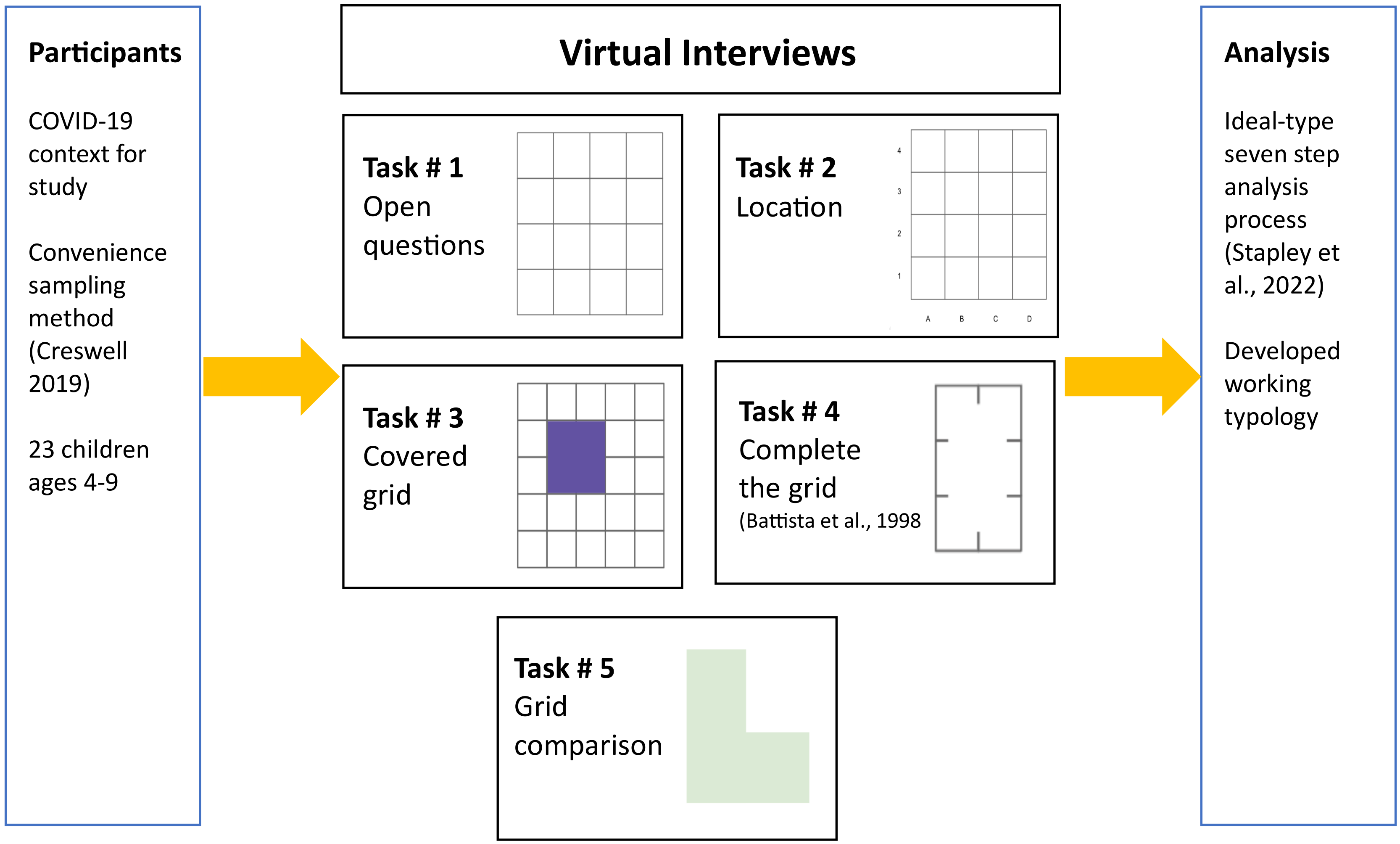
Appendix A: Research design summary illustration



Appendix B. COVID-19 Protocols

Participants and the researcher will confirm they have taken an online COVID-19 self-assessment before the materials are dropped off (see protocol below). The researcher will ensure the box of materials is dropped off no later than three days prior to the scheduled interview time.

The researcher has designed a protocol for distributing physical materials to minimize risk of spread of COVID-19.

Protocol for distribution of physical materials:

- All paper materials, including sticky notes and transparencies, are only used once, fresh sets are printed for each participating family
- Plastic tiles are disinfected by the researcher after each use
- Laptop, iPad and iPad stand are disinfected by the researcher after each use
- The researcher will wear a mask and gloves while packaging items in a box
- A box of materials is dropped off at the participants doorstep, or a convenient outdoor location, at an agreed upon time. This will not require any physical interaction between the participant and the researcher. A mask and gloves will be worn by the researcher when dropping off the box.
- After the interview is complete participants return all the items to the box and let the researcher know that they are ready for pick up. Participants leave the box on their doorstep, or a convenient outdoor location, at an agreed upon time and the items will be picked up by the researcher. This will not require any physical interaction between the participant and the researcher. The researcher will wear a mask and gloves while picking up the box.
- After the researcher picks up the box all items will be quarantined for 72 hours before being disinfected and prepared for loaning out to the next participant.
- All participants are asked to complete a COVID-19 screening prior to arranging pick-up and drop-off of materials. Participants complete an online screening using the Government of Ontario's self-assessment tool: https://covid-19.ontario.ca/self-assessment/. Participants email the researcher to confirm they have completed the online COVID-19 self-assessment prior to pick-up and drop-off of materials.
- The researcher will complete a COVID-19 screening prior to picking up and dropping off materials. The researcher will complete the screening online using the Government of Ontario's self-assessment tool: https://covid-19.ontario.ca/self-assessment/. The researcher will email the participants to confirm they have completed the online COVID-19 self-assessment prior to pick-up and drop-off of materials.
- Contact information of each participant will be held securely by the researcher to facilitate COVID-19 contact tracing if needed

Once the interviews are completed the parents are asked to return all the materials used in each task to their original envelopes to place them back inside the box. The researcher will email participants to arrange a time and location to pick up the materials. Participants will be reminded to complete a COVID-19 screening prior to the pick-up of materials. They will complete the screening online using the Government of Ontario's self-assessment tool: https://covid-19.ontario.ca/self-assessment/. Participants will email the researcher to confirm they have completed the online COVID-19 self-assessment. The researcher will take the same self-assessment tool online and confirm this process with participants via email prior to picking up the materials.

Appendix C. Interview Script

Interview Script:

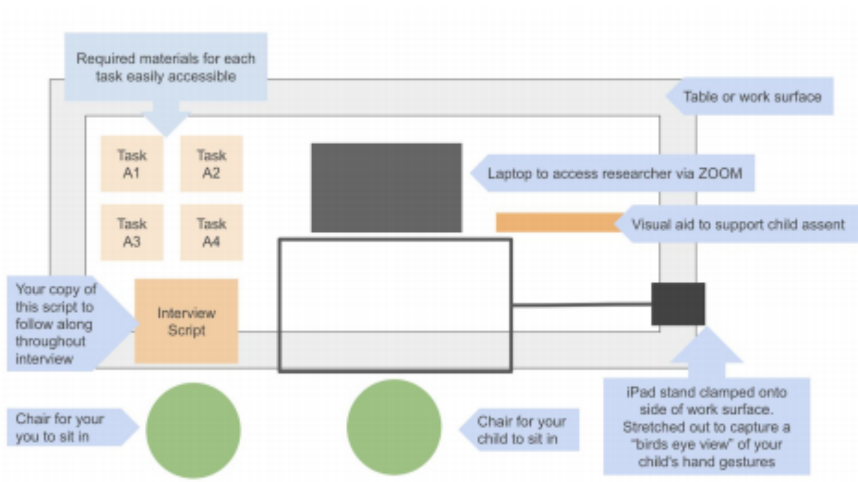
Hello!
Thank you so much for taking the time to participate in this study. If you have any questions at all as you read through the materials please don't hesitate to ask. If you would prefer to go through these instructions over the phone or over ZOOM I would be happy to meet with you at your convenience.

Please find below instructions on:

- Setting up a work space for the interview
- Setting up audio and video recording equipment
- Talking to your child about participation
- An interviewers script that details each task (the bolded blue sections highlight your role)
- How to repackage materials

Setting up a work space for the interview:
Your child will be working through four math tasks at each twenty minute interview. Audio and video recording will be needed to capture their verbal responses. Try to find a quiet space in your home that will allow for minimal background noise to be captured on the recording. A table or work surface will be needed as students need a space to work. At key points in the interview you will be asked to provide your child with the necessary materials (e.g., plastic tiles, task cards). These materials are labelled and found in the box of physical materials. Note the system for labelling materials. There are five tasks in the Interview; they are labeled B.1, B.2, B.3, B.4., B.5. Within each task there may be multiple questions in which case the materials are labelled B.1.1, B.1.2, B.1.3 etc to correspond with the questions within a task. You will want to set them out so you can access them quickly and easily throughout the interview.

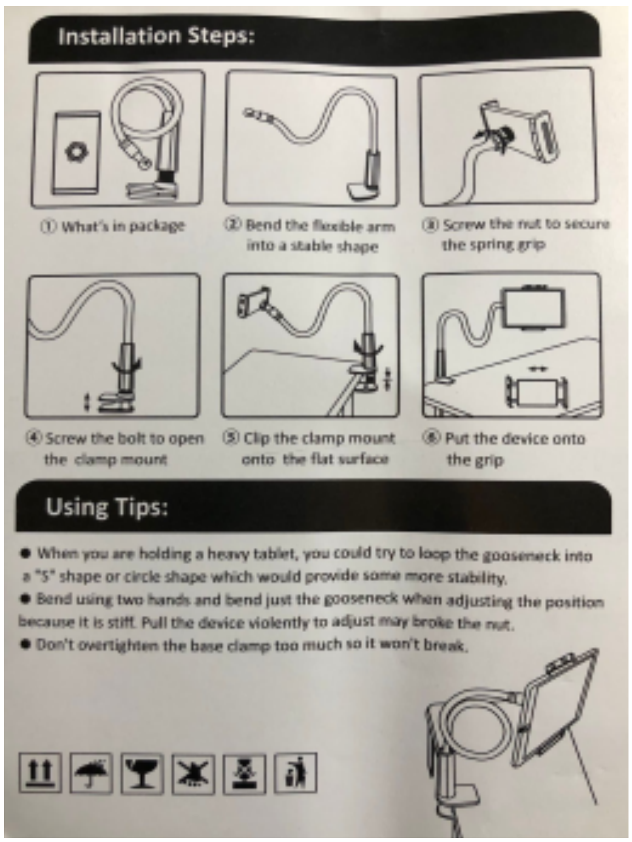
Here is a diagram of how you might organize the work space:



<u>Setting up audio and video recording equipment:</u>

The laptop provided comes with the ZOOM application pre-installed. If you are choosing to use your own iPad or laptop you can download the ZOOM application here: [ZOOM Download](). On the interview day you will receive a ZOOM link via email from the researcher. Once you click the link in the email you will enter the ZOOM conference. The researcher will help you test the audio and video before beginning the interview. The researcher will record the meeting on their computer; you only need to enter the ZOOM conference.

You will also need to set up the iPad stand. Here are the product instructions for using the iPad stand:



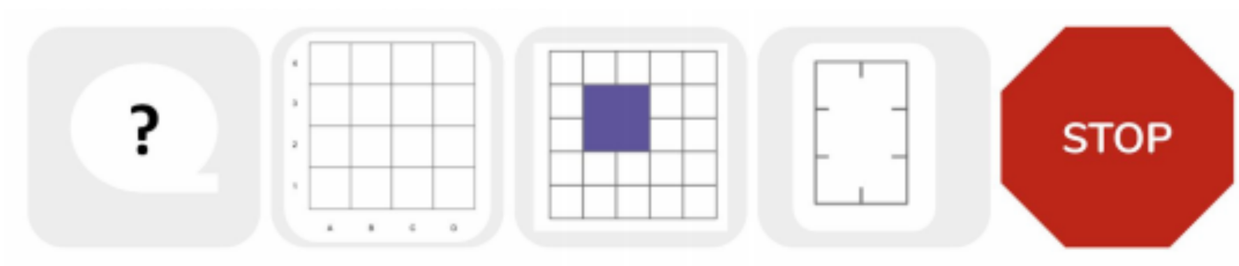
We hope to capture all of your child's hand gestures as they engage with the tasks. It is better to go too wide as opposed to too tight when positioning the iPad. Use the camera setting on the iPad to record video. Do a quick trial video using your own hands to ensure you are capturing the work area. Make sure the iPad does not obstruct the child's view of the researcher or materials. It may need to be a little off center to achieve this. Throughout the interview there may be times where your child leans over and therefore blocks the iPads view, a slight side angle could help avoid this. We can work through this together over ZOOM or over the phone if you have difficulties setting up the equipment.

Talking to your child about participation:

It is important that your child decides for themselves that they would like to participate in this study. Interviews can be paused, stopped or re-scheduled at any time that you or your child wish. In order to help your child understand what we are hoping to do together we have written an explanation that we will read to your child at the start of the interview.

Here is a copy of what the researcher will say at the start of the interview:
(note: please have the visual aid to support your child's assent in front of the child as the researcher reads this explanation)

Example of explanation:
Hi my name is Jess! I love to play math games. I made up some of my own. Would you want to play some with me? I am trying to learn what you are thinking when you play so I'm going to ask you lots of questions. Your (parent name) has a tracker that shows the games we can play together. Look, there are four, let's count them, 1, 2, 3, 4. There is a stop sign too. If you don't want to play anymore you can put your finger on the stop sign at any time and we will stop! Do you have any questions for me? Do you want to get started?



Task Details:

Please find below details of what each task entails and how it will unfold. You will see a coding system to organize each task, e.g., Interview A has four tasks: they are coded B.1, B.2, B.3 and B.4. Within a task there might be multiple questions: they are coded A.1.1, A.1.2, B.1.2, B.1.3 etc. The necessary materials are coded to match the task using this system. Note that the researcher's speech is italicized. Instructions for your support are in bolded blue font. You will want to read through these tasks ahead of the interview to ensure you understand how you support each task. It is important that you refrain from re-phrasing instructions or giving additional prompts. The researcher has planned prompts to support your child if they get stuck. If you have any questions about a task or think a task should be omitted please let the researcher know at the start of the interview. If you would like to talk through these instructions with the researcher ahead of time please don't hesitate to ask!

How to repackage materials:

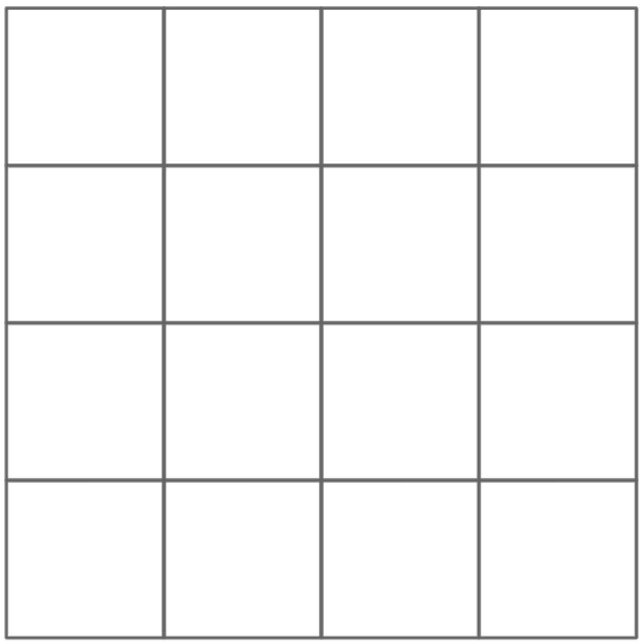
The materials for each task can be returned to their original envelopes and placed back inside the box. Once both interviews have been completed the researcher will email you to arrange a time and location to pick up the box. A time will be arranged that is convenient for you. You are asked to complete a COVID-19 screening prior to the pick-up of materials. You complete the screening online using the Government of Ontario's self-assessment tool: https://covid-19.ontario.ca/self-assessment/. You will be asked to email the researcher to confirm you have completed the online COVID-19 self-assessment. The researcher will take the same self-assessment tool online and confirm this process with you via email prior to picking up the materials.

Thank you so much for taking the time to participate in this study. Your participation and your child's participation are a generous donation to the research in mathematics education. If you have any questions please do not hesitate to reach out.

Thank you,
Jessica Bodnar

Appendix D. Task 1: Open questions

B.1. Open questions about the grid.



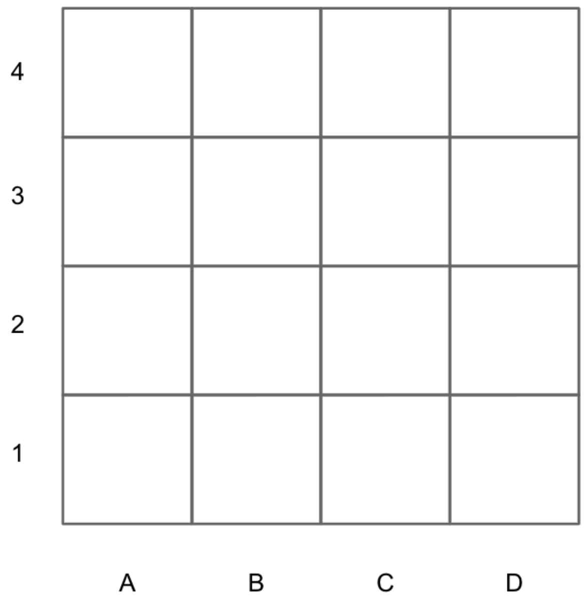
Materials Needed: Print out of grid labelled B.1

Students will be asked some general open ended to questions to get them talking about the grid. The first question will ask the child what the grid reminds them of, this is to get a broad sense of their relationship to the spatial structure. The next question will ask them to be more specific and will direct them to look at the parts that make up the grid. Finally students will be asked to generalize their understanding by considering some possible uses for the grid.

Parent places grid labelled B.1 in front of the child
Researcher: *"What does this remind you of?"*
(pause for student to respond)
Researcher: *"Look closely, what parts do you see?"*
(pause for student to respond)
Researcher: *"What could you use this for?"*
(pause for student to respond)
Parent removes grid labelled B.1
End of task

Appendix E. Task 2: Location on grids

B.2. Location on grids



Materials Needed: Location grid labelled B.1, plastic tiles

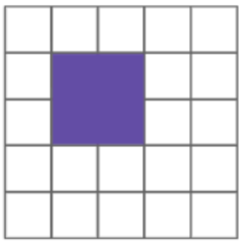
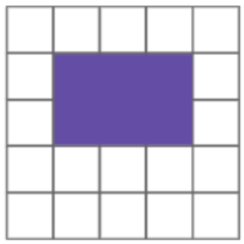
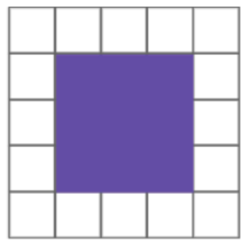
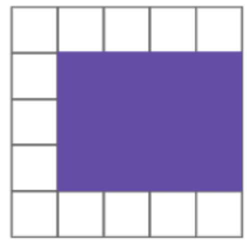
This task asks students to place a plastic tile on specific locations on a labelled grid. The labels on the grid show the letters A, B, C, D along the x-axis. The y-axis is labelled with the numbers 1, 2, 3, 4. The researcher begins by directing the students attention to these labels. This will take some coordination between the parent and researcher. The coordination is between the researcher's words and the parents gestures that serve to highlight the specific area of the grid for students. For example when the researcher counts out each of the horizontal rows the parent will use a smooth sweeping motion along the row. This motion helps children see that the numbers labelling the row apply all along the entire row. The same coordination is required for highlighting the columns. The direction of the parents gesture is noted as well (e.g., bottom to top, left to right). The description of the labeled grid only happens at the beginning to set up the task. All remaining questions related to the task involve students placing a tile on a designated spot on the grid. Here is an example of what the grid will look like:

Researcher: “*Here is a grid. It has four columns*”
Parent uses a smooth sweeping gesture running their finger along each column from bottom to the top and says aloud A, B, C, D to highlight each column
Researcher: “*and it has four rows*”

Parent uses a smooth sweeping gesture running their finger along each row from left to right and says aloud 1, 2, 3, 4 to highlight each row
Researcher: “*Place a tile on C2”*
Parent hands child a plastic tile
(pause while student places tile)
Researcher: *“Thank you”*
Parent removes tile from the grid
Researcher: “*Place a tile on D3”*
Parent hands child a plastic tile
(pause while student places tile)
Researcher: *“Thank you”*
Parent removes tile from the grid
Researcher: *“Now I’m going to place a tile down”*
Parent places a tile on B3
Researcher: *“What’s the name of my spot? Use the letters and numbers.”*
(pause while student responds)
Researcher: *“Thank you”*
Parent removes tile from the grid
Researcher: *“Now I’m going to place another tile down”*
Parent places a tile on C2
Researcher: *“What’s the name of my spot? Use the letters and numbers.”*
(pause while student responds)
Researcher: *“Thank you”*
Parent removes tile from the grid
End of task

Appendix F. Task 3: Covered grid task

<u>B.3. Covered grid "blanket" task</u>
Materials Needed: Covered grid cards labelled B.3.1, B.3.2, B.3.3, B.3.4.

B.3.1	B.3.2	B.3.3	B.3.4
			

This task shows students a series of 5x5 grids. On the 5x5 grids there is a rectangular section that is coloured purple covering up the grid lines. Students are told that this smaller purple rectangle is a "blanket". They are asked to determine how many of the grid squares have been covered by the blanket. They are also prompted to explain their rationale for determining the number of squares. There are four questions within this task. Each blanket card has been intentionally sequenced. Here is an example of the sequence of blanket sizes students will encounter:

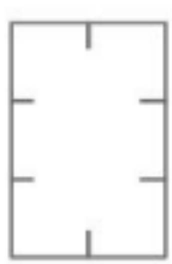
Parent places blanket grid card B.3.1 in front of child
Researcher says: *"Here is a 5x5 grid. There are 5 columns..."*
Parent uses a sweeping gesture from bottom to top to indicate each column
Researcher: "*...and there are 5 rows"*
Parent uses a sweeping gesture from left to right to indicate each row
Researcher: *"There is a blanket covering some of the squares on this grid. How many squares do you think are hiding under the blanket?"*
(pause for student response)
Researcher: "*How did you decide it was ___?"*
Parent removes grid B.3.1 and places B.3.2 in front of child
Researcher: "*Now how many squares are hiding under the blanket?"*
(pause for student response)
Researcher: "*How did you decide it was ___?"*
Parent removes grid B.3.2 and places B.3.3 in front of child
Researcher: "*Now how many squares are hiding under the blanket?"*
(pause for student response)
Researcher: "*How did you decide it was ___?"*
Parent removes grid B.3.4 and places B.3.5 in front of child
End of task

Appendix G. Task 4: Complete the grid

B.4. Complete the grid (Battista et. al., 1998) \
Materials Needed: Complete the grid cards B.4.1, B.4.2, B.4.3, pencil, plastic tiles

Students are presented with an image of a grid that is missing the interior grid lines. Some indicators are visible to help students see where the grid line should extend. The grid has been scaled so that one cell of the grid covers a 1inch by 1inch area. The plastic tiles are 1inch by 1inch and will cover one grid cell completely. Students are first asked to estimate how many plastic tiles they think it would take to cover the entire grid. They are shown one plastic tile and prompted to use this as a visual point of reference. The researcher then asks the student to complete the grid, a prompt designed to see if students can visually extend the grid lines across the grid. If the students drawing matches their prediction the researcher will move onto the next question. If there is a mismatch between students prediction and drawing the researcher will prompt the child to check their thinking by covering the grid with the plastic tiles.

Here is an example of a grid with missing grid lines:



Parent places grid B.4.1 in front of child

Researcher: "*Here is another grid, but look this one is missing some parts. How many square tiles would you need to cover this grid completely?"*

Parent holds up one square tile, if student tries to take the square tile the parent will say "try to picture it in your mind"

Researcher: Repeats prompt if needed "*How many square tiles would you need to cover this grid completely?"* (pause for student response)

Researcher: "*Can you use a pencil to finish the grid?*

Parent hands student a pencil

Researcher: Repeats prompt if needed "*Can you finish the grid?*
(pause for student response)

Researcher: "*How many squares do you have?"*

If the response is correct, move to the next size grid.

If incorrect researcher asks: "*Let's check with the tiles"*

Parent hands student tiles

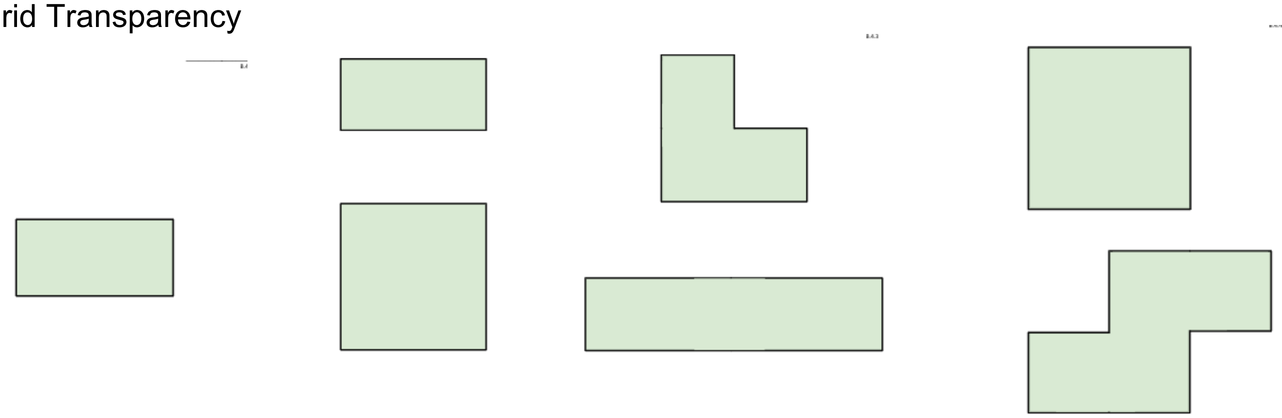
(pause for student to cover grid)

Researcher: "*How many tiles do you have?*
(pause for student response)

Researcher: "*What did you notice?*
(pause for student response)
Parent removes B.4.1 and places B.4.2 in front of child
Researcher: Repeats prompt if needed "*How many square tiles would you need to cover this grid completely?"* (pause for student response)
Researcher: "*Can you use a pencil to finish the grid?*
Parent hands student a pencil
Researcher: Repeats prompt if needed "*Can you finish the grid?*
(pause for student response)
If the response is correct, move to the next size grid.
If incorrect researcher asks: "*Let's check with the tiles"*
Parent hands student tiles
(pause for student to cover grid)
Researcher: "*How many tiles do you have?*
(pause for student response)
Researcher: "*What did you notice?*
(pause for student response)
Parent removes B.4.2 and places B.4.3 in front of child
Researcher: Repeats prompt if needed "*How many square tiles would you need to cover this grid completely?"* (pause for student response)
Researcher: "*Can you use a pencil to finish the grid?*
Parent hands student a pencil
Researcher: Repeats prompt if needed "*Can you finish the grid?*
(pause for student response)
If the response is correct, move to the next size grid.
If incorrect researcher asks: "*Let's check with the tiles"*
Parent hands student tiles
(pause for student to cover grid)
Parent removes B.4.3
End of task

Appendix H. Task 5: Comparing with transparent grid

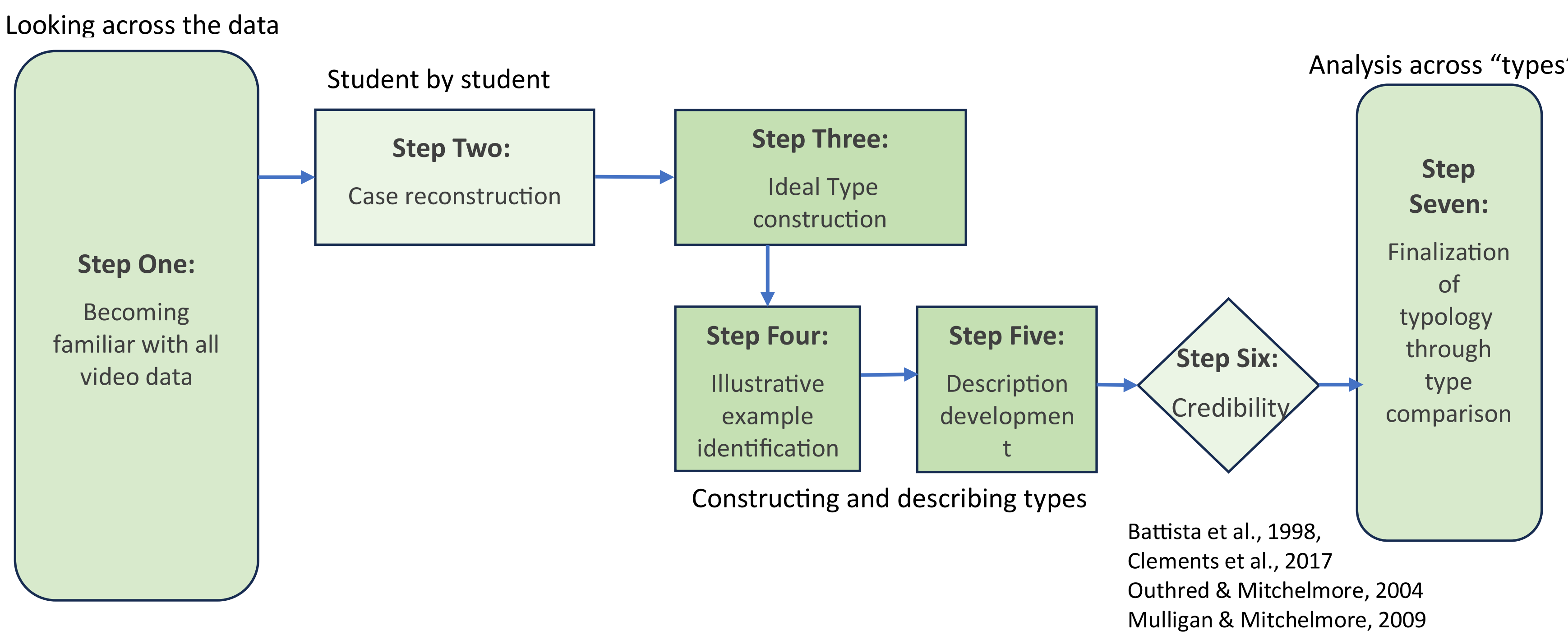
B.5 Grid Comparison Task
Materials Needed: Grid Comparison Task Cards B.5.1, B.5.2, B.5.3, B.5.4, and Grid Transparency



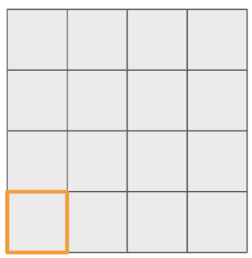
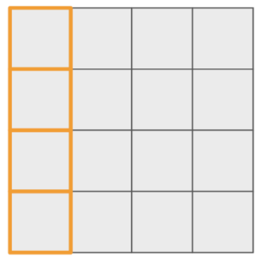
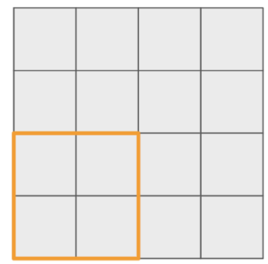
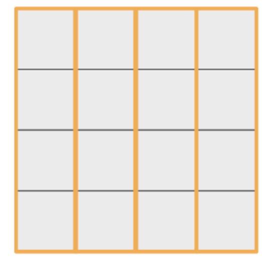
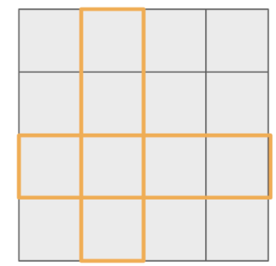
This task is designed to prompt students to use the grid to make a comparison between the different gardens. Students may or may not overlay a grid transparency onto the shapes to help them compare. The researchers first question is designed as a practice question to help students understand the task. The remaining questions ask students to compare the gardens. Students may choose to use the grid to help with this comparison or they may use other invented strategies (e.g., estimation, mental rotation).

Parent places task card B.5.1 in front of child
Researcher: *"This is a drawing of a garden. How could you tell how big the garden is?"*
Parent lays out grid transparency which remains in child's reach for remainder of task
Researcher: repeats prompt *"How big is the garden?"*
Parent removes B.5.1 and places B.5.2 in front of child
Researcher: *"Here are two different gardens. Point to which garden you think is bigger?"*
Researcher: *"How do you know that one is bigger?"*
Parent removes B.5.2 and places task card B.5.3 in front of child
Researcher: *"Point to which garden you think is bigger?"*
Researcher: *"How do you know that one is bigger?"*
Parent places task card B.5.3 and places B.5.4 in front of child
Researcher: *"Point to which garden is bigger?"*
Researcher: *"How do you know that one is bigger?"*
End of task

Appendix I. Summary diagram of the ideal-type analysis process used in this thesis (Stapley et al., 2022)



Appendix J. Working typology of students spatial structuring of square grids

1. Single Cell Structuring	2. Partial Unit Building	3. Whole Figure and Parts-of-Figure Noticing	4. Composite Unit Structuring	5. Coordinated Structuring
				
Sees each cell as an individual unit with its own structure. The collection of cells are not bound by an overall system. The relationship between cells is non-continuous.	Sees an extended collection of the individual cells within rows and columns. Recognizes a combined relationship between cells in row or cells in a column.	Perceives the whole grid as well as its component parts such as rows and columns. Composes and decomposes shape isolating geometric figures within the grid. Conveys a sense of coordinated space.	Unitzes (groups) features of the grid to make sense of the larger structure. Isolates a unit (row or column) made up of individual units. Sees copies of the composite unit within the grid. Coordinates quantities (e.g., counts rows of four individual units as 1, 2, 3 4).	Perceives the coordination of multiple dimensions simultaneously. Recognizes a generalized relationship between rows and columns and their intersections.
- Taps each cell with pointer finger - May double tap when counting especially on cells whose structure is less discernible (e.g., corners, interior) - May count along a pathway supported by the visible structure of the perimeter of shape (e.g., spirals around towards interior where cells are less distinguishable) - May count all along row or column in a back and forth weaving pathway	- Fluid motion shows sequenced collection of individual units - May run finger along grid lines to help isolate individuals cells - May run finger along grid lines to partition space into cells - May run finger along centre of row/column to highlight collection - May trace an extended range of cells or grid line	- Frames geometric figure (usually with both hands or tracing of the perimeter) - Uses knowledge of geometric figures to make sense of spatial structure of the grid - Breaks down overall grid into recognizable geometric figures (mentally transforms, covers or copies figures) - Repeating geometric “parts” may or may not fill grid completely (e.g., square and half a square or square and 2 more in 2x6)	- Tap indicator when counting composite row/column units (e.g., whole hand marker, finger signifier, scaling pincher) - May use a scaling pincher gesture to measure the size of the unit being copied - Beat often rythmic to mark the unit counting	- Motions in both directions simultaneously to highlight row/column relationship - May use more robust gestures (e.g., both hands, use of fluid motion)